Robust, Semi-Intelligible Isabelle Proofs from ATP Proofs

Steffen Juilf Smolka
Jasmin Christian Blanchette

ITPs vs. ATPs

ITPs

Isabelle

well suited for
large formalizations

but

require intensive
manual labor

ATPs

SPASS

Vampire

LEO-II

E

SATALLAX

fully automatic

but

no proof
management

ITPs

ATPs

well suited for
large formalizations

but

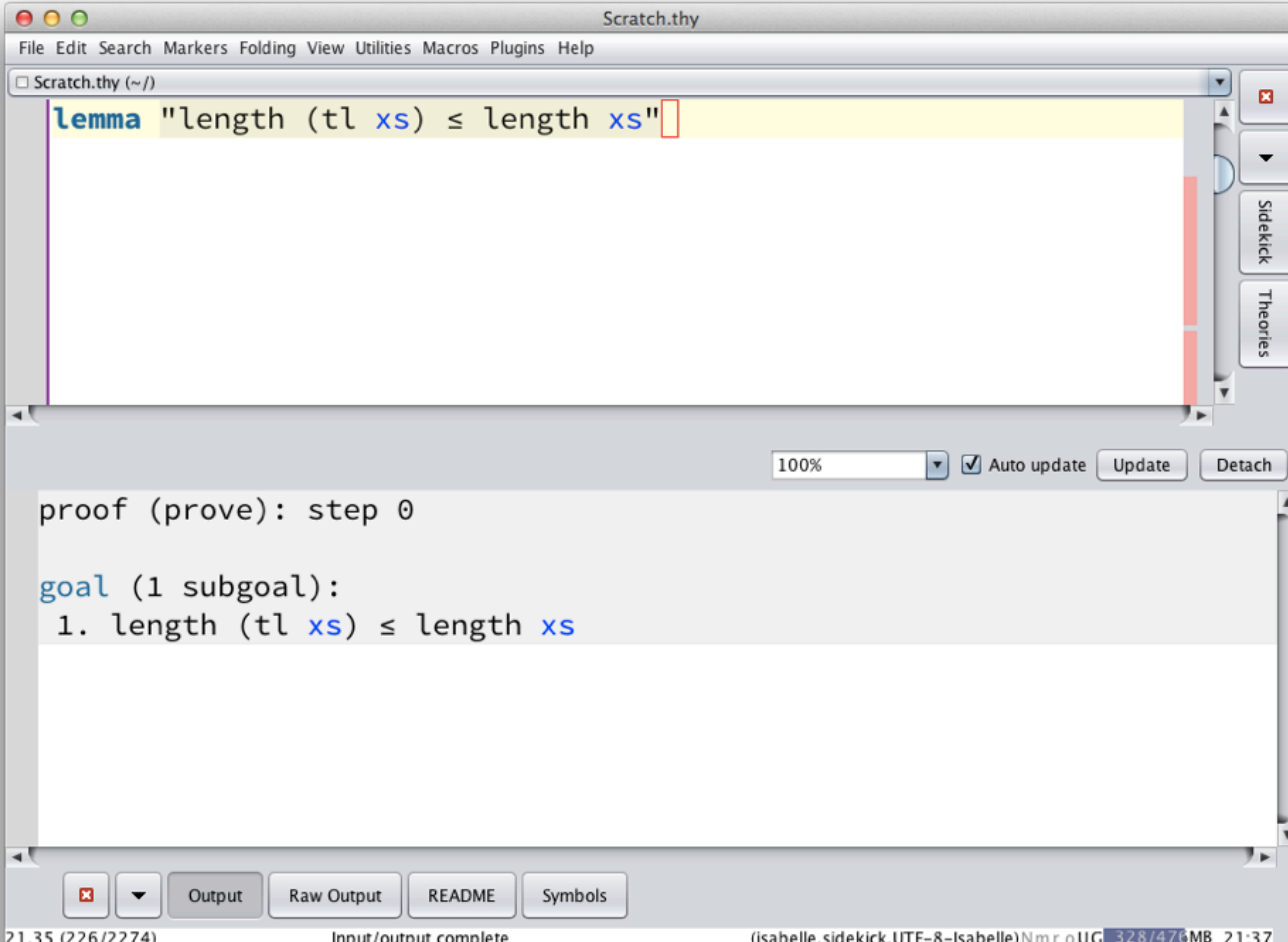
require intensive
manual labor

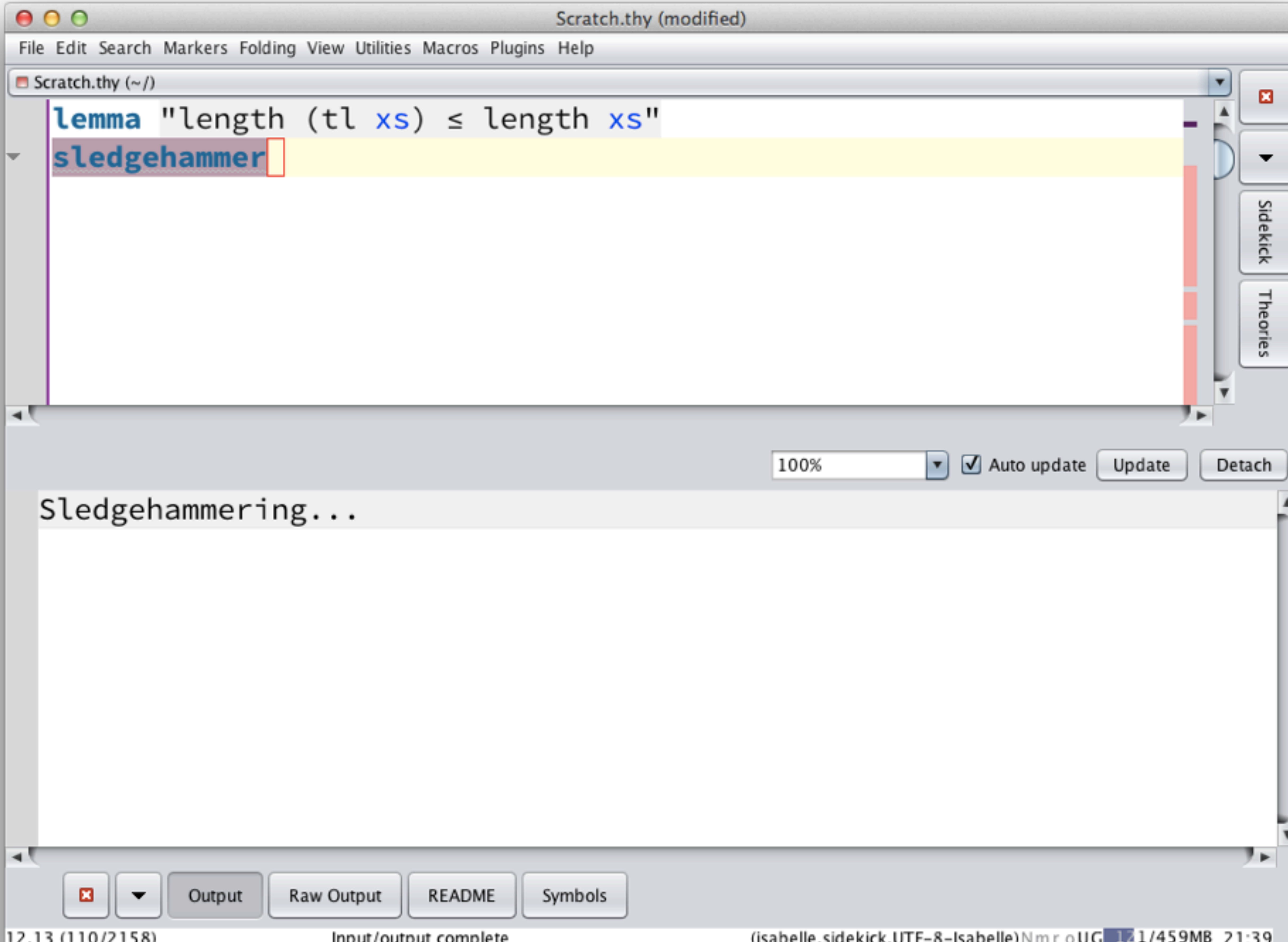
**Sledge-
hammer**

fully automatic

but

no proof
management


Scratch.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
Scratch.thy (~/)
lemma "length (tl xs) ≤ length xs"
Sidekick
Theories
100%
Auto update
Update
Detach
proof (prove): step 0
goal (1 subgoal):
1. length (tl xs) ≤ length xs
Output
Raw Output
README
Symbols
21,35 (226/2274)
Input/output complete
(isabelle,sidekick,UTF-8-Isabelle)
21:37


Scratch.thy (modified)
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lemma "length (tl xs) ≤ length xs"
sledgehammer
Sidekick
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Sledgehammering...
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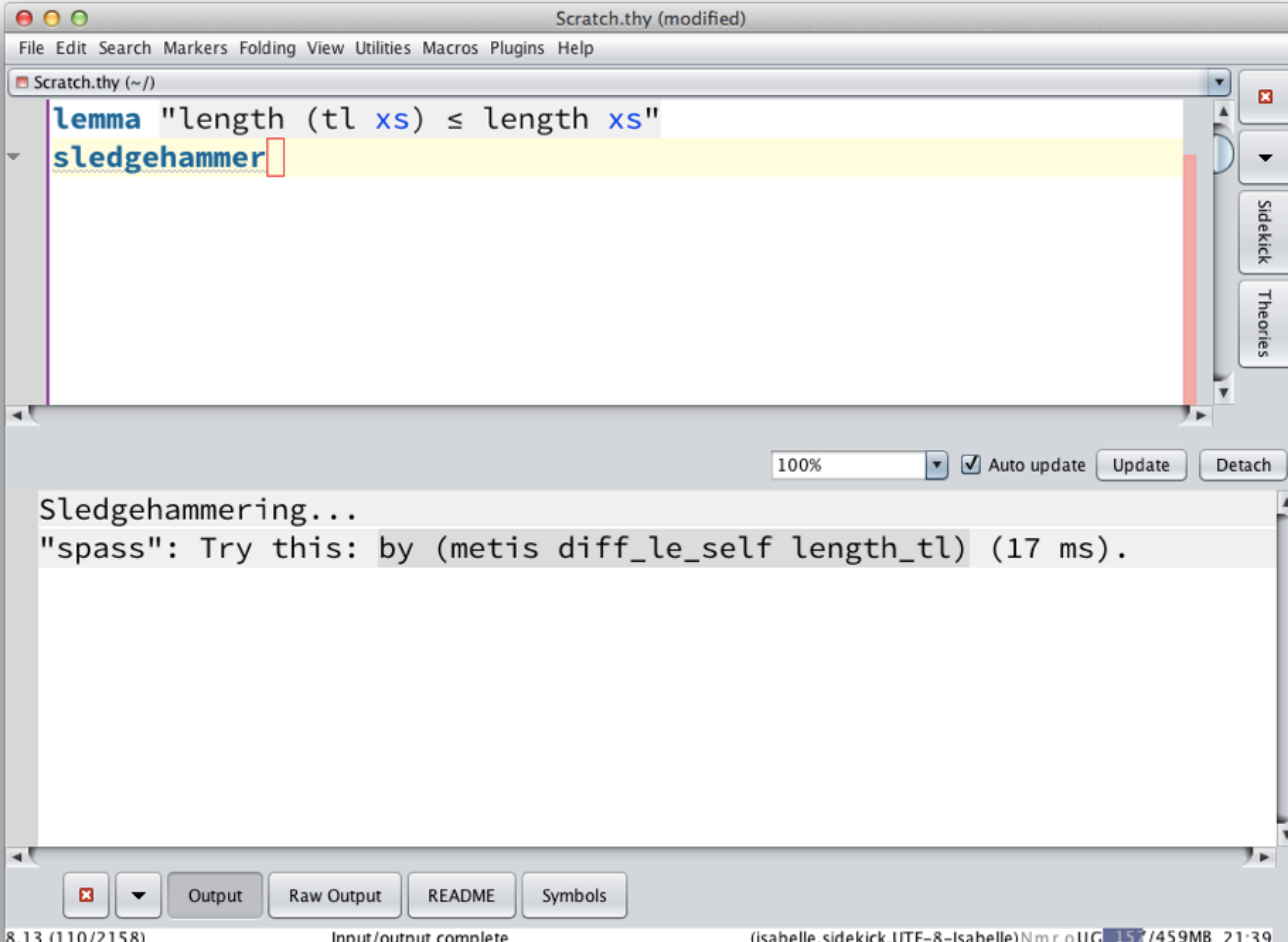
README
Symbols
12,13 (110/2158)
Input/output complete
(isabelle,sidekick,UTF-8-Isabelle)
1/459MB 21:39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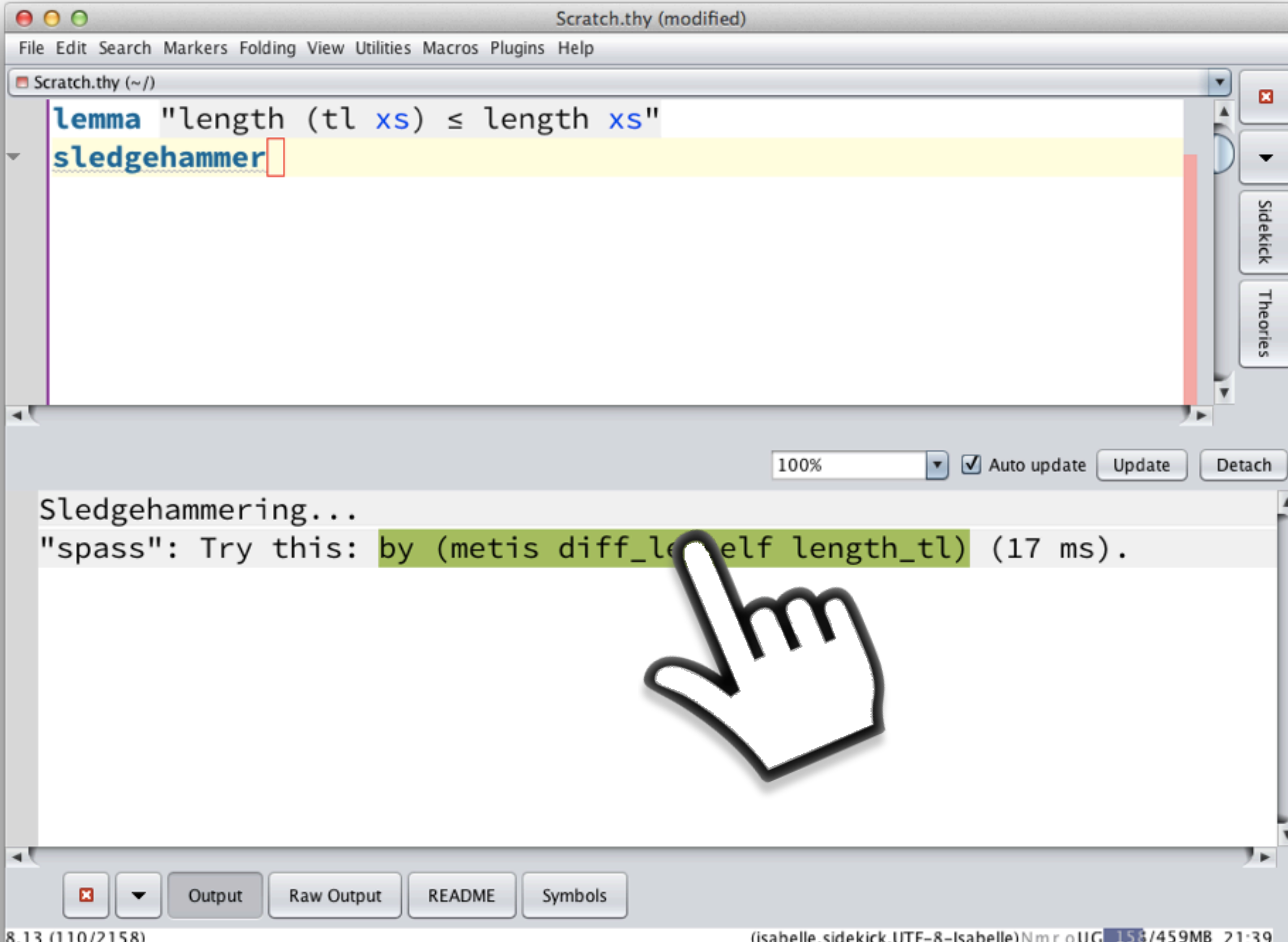
lemma "length (tl xs) ≤ length xs"
sledgehammer
Sidekick
Theories
100%
Auto update
Update
Detach
Sledgehammering...
"spass": Try this: by (metis diff_le_self length_tl) (17 ms).
Output
Raw Output
README
Symbols
8,13 (110/2158)
Input/output complete
(isabelle,sidekick,UTF-8-Isabelle)
/459MB 21:39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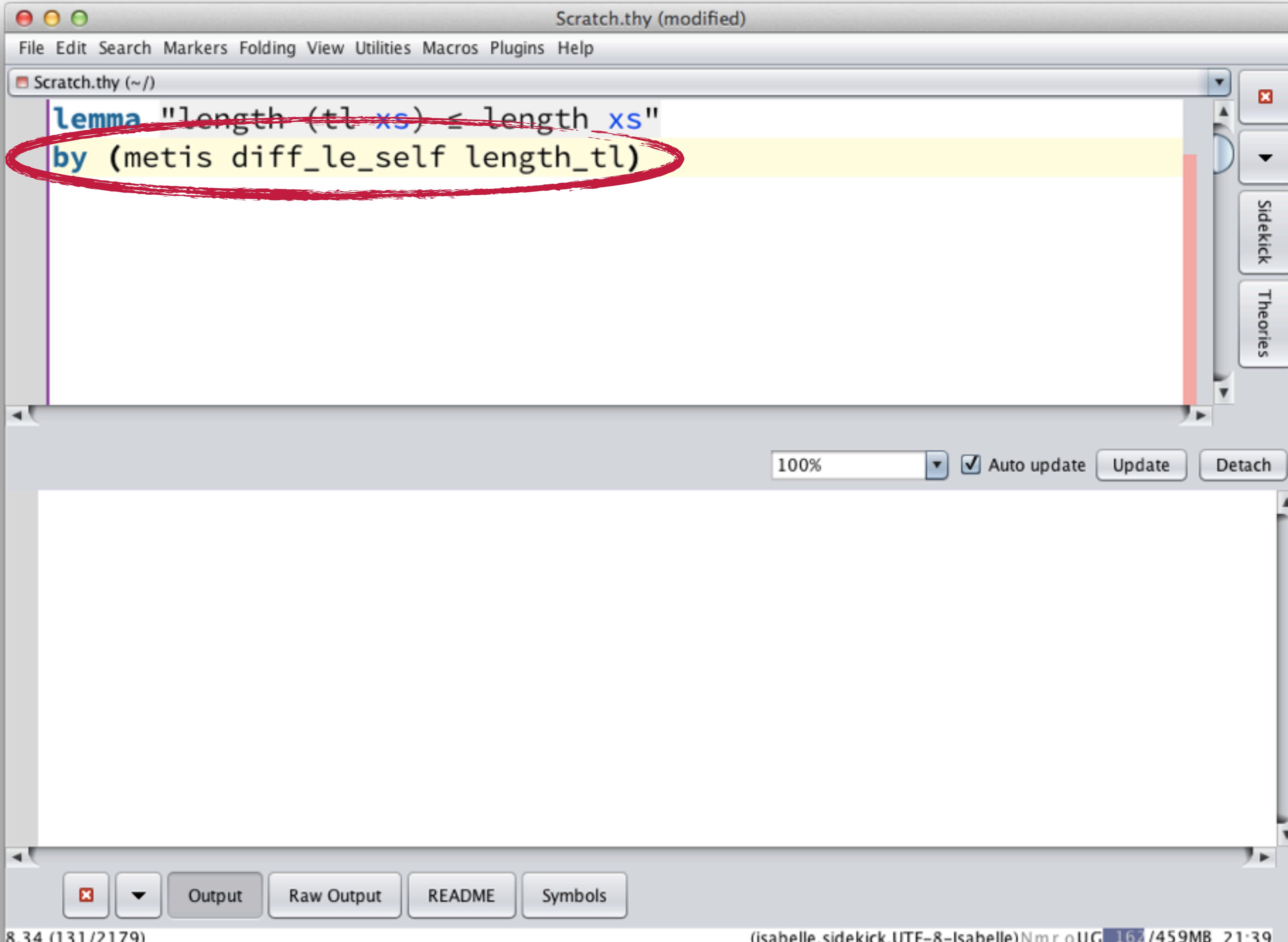
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8,34 (131/2179)
(isabelle,sidekick,UTF-8-Isabelle)Nmr oUG 162/459MB 21:39

Exploit ATPs,

but don’t trust them.

LCF Principle (Robin Milner):

Have all proofs checked by the inference kernel.

$\Longrightarrow$ ATP proofs must be **reconstructed** in Isabelle.

Approach A: Metis One-Liners

```
lemma "length (tl xs) ≤ length xs"
by (metis diff_le_self length_tl)
```

proof method

lemmas

Approach A: Metis One-Liners

```
lemma "length (tl xs) ≤ length xs"
by (metis diff_le_self length_tl)
```

proof method **lemmas**

external ATPs: find proof given 100s of facts

Metis: re-find proof given only necessary facts

Approach A: Metis One-Liners

```
lemma "length (tl xs) ≤ length xs"
by (metis diff_le_self length_tl)
```

proof method

lemmas

external ATPs: find proof given 100s of facts

Metis: re-find proof given only necessary facts

+ usually fast and reliable
+ lightweight

- cryptic
- sometimes slow (several seconds)
- on avg. 5% > 30 seconds

Approach B: Detailed Isar Proofs

```
lemma "length (tl xs) ≤ length xs"
proof -
  have "⋀x1 x2. (x1::nat) - x2 - x1 = 0 - x2"
    by (metis comm_monoid_diff_class.diff_cancel diff_right_commute)
  hence "length xs - 1 - length xs = 0"
    by (metis zero_diff)
  hence "length xs - 1 ≤ length xs"
    by (metis diff_is_0_eq)
  thus "length (tl xs) ≤ length xs"
    by (metis length_tl)
qed
```

Approach B: Detailed Isar Proofs

```
lemma "length (tl xs) ≤ length xs"
proof -
  have "⋀x1 x2. (x1::nat) - x2 - x1 = 0 - x2"
    by (metis comm_monoid_diff_class.diff_cancel diff_right_commute)
  hence "length xs - 1 - length xs = 0"
    by (metis zero_diff)
  hence "length xs - 1 ≤ length xs"
    by (metis diff_is_0_eq)
  thus "length (tl xs) ≤ length xs"
    by (metis length_tl)
qed
```

+ faster than one-liners

+ 100% reconstruction (in principle)

+ self-explanatory

- technically more challenging

Challenge 1:

Resolution proofs are by contradiction

"sin against mathematical exposition" (Knuth et al. 1989)

→ Jasmin Blanchette

Challenge 2:

Skolemization - introduce new symbols during proof

Challenge 3:

Type Annotations - make Isabelle understand its own output

Challenge 4:

Preplay & Compression - test and optimize proofs

Challenge 2:

Skolemization

$$\frac{\forall X.\ \exists Y.\ p(X, Y)}{\forall X.\ p(X, \mathbf{y}(X))}\quad \text{Skolemization}$$

Signature is extended

$$\frac{\forall X.\ \exists Y.\ p(X,\ Y)}{\forall X.\ p(X,\ \mathbf{y}(X))}\quad \text{Skolemization}$$

Signature is extended

$$\frac{\forall X.\ \exists Y.\ p(X,\ Y)}{\exists \mathbf{y}.\ \forall X.\ p(X,\ \mathbf{y}(X))}\quad \text{Ax. of Choice}$$

$$\frac{\forall X.\ \exists Y.\ p(X, Y)}{\forall X.\ p(X, \mathbf{y}(X))}\ \text{Skolemization}$$

Signature is extended

$$\frac{\forall X.\ \exists Y.\ p(X, Y)}{\exists \mathbf{y}.\ \forall X.\ p(X, \mathbf{y}(X))}\ \text{Ax. of Choice}$$

Signature is extended

$$\textit{obtain}\ \mathbf{y}\ \textit{where}\ \forall X.\ p(X, \mathbf{y}(X))$$

<steps with **reduced** sig.>

$$\frac{\forall X.\ \exists Y.\ p(X,\ Y)}{\exists \mathbf{y}.\ \forall X.\ p(X,\ \mathbf{y}(X))}\ \text{Ax. of Choice}$$

<steps with **extended** sig.>

<steps with **extended** sig.>

$$\frac{\exists y.\ \forall X.\ p(X,\ y(X))}{\forall X.\ \exists Y.\ p(X,\ Y)}\ \text{Ax. of Choice}$$

<steps with **reduced** sig.>

<steps with **extended** sig.>

$$\frac{\forall \mathbf{y}.\ \exists X.\ \neg p(X,\ \mathbf{y}(X))}{\exists X.\ \forall Y.\ \neg p(X,\ Y)}\ \text{Ax. of Choice}$$

<steps with **reduced** sig.>

<steps with **extended** sig.>

$$\frac{\forall y.\ \exists X.\ \neg p(X,\ y(X))}{\exists X.\ \forall Y.\ \neg p(X,\ Y)}$$

Contrap. of
Ax. of Choice

<steps with **reduced** sig.>

<steps with **extended** sig.>

$$\frac{\forall y.\ \exists X.\ \neg p(X,\ y(X))}{\exists X.\ \forall Y.\ \neg p(X,\ Y)}$$

Contrap. of Ax. of Choice

<steps with **reduced** sig.>

{ fix **y**

<steps with **extended** sig.>

have $\exists X.\ \neg p(X,\ y(X))$ }

hence $\exists X.\ \forall Y.\ \neg p(X,\ Y)$

<steps with **reduced** sig.>

Challenge 3:

Type Annotations

Make Isabelle understand its own output

$$2^{\texttt{nat}} \;+^{\texttt{nat}\to\texttt{nat}\to\texttt{nat}}\; 2^{\texttt{nat}} \;=^{\texttt{nat}\to\texttt{nat}\to\texttt{bool}}\; 4^{\texttt{nat}}$$

$$\Big\downarrow\ \text{print}$$

$$2 + 2 = 4$$

$$2^{\texttt{nat}} \;+^{\texttt{nat}\to\texttt{nat}\to\texttt{nat}}\; 2^{\texttt{nat}} \;=^{\texttt{nat}\to\texttt{nat}\to\texttt{bool}}\; 4^{\texttt{nat}}$$

$$\Big\downarrow \; \text{print}$$

$$2 + 2 = 4$$

$$\Big\downarrow \; \text{parse}$$

$$2^{\alpha} \;+^{\alpha\to\alpha\to\alpha}\; 2^{\alpha} \;=^{\alpha\to\alpha\to\texttt{bool}}\; 4^{\alpha}$$

where $\alpha : \texttt{numeral}$

Un-provable

$$\mathbf{2}^{\mathtt{nat}}\ \ \mathbf{+}^{\mathtt{nat\to nat\to nat}}\ \ \mathbf{2}^{\mathtt{nat}}\ \ \mathbf{=}^{\mathtt{nat\to nat\to bool}}\ \ \mathbf{4}^{\mathtt{nat}}$$

$$\Big\downarrow\ \text{print}$$

(**2**:nat) (**+**:nat→nat→nat) (**2**:nat)
(**=**:nat→nat→bool) (**4**:nat)

$$\Big\downarrow\ \text{parse}$$

$$\mathbf{2}^{\mathtt{nat}}\ \ \mathbf{+}^{\mathtt{nat\to nat\to nat}}\ \ \mathbf{2}^{\mathtt{nat}}\ \ \mathbf{=}^{\mathtt{nat\to nat\to bool}}\ \ \mathbf{4}^{\mathtt{nat}}$$

$$2^{\mathtt{nat}} \;+^{\mathtt{nat\to nat\to nat}}\; 2^{\mathtt{nat}} \;=^{\mathtt{nat\to nat\to bool}}\; 4^{\mathtt{nat}}$$

$$\big\downarrow \text{print}$$

$$2 + 2 = 4$$

$$\big\downarrow \text{parse}$$

$$2^{\alpha} \;+^{\alpha\to\alpha\to\alpha}\; 2^{\alpha} \;=^{\alpha\to\alpha\to \mathtt{bool}}\; 4^{\alpha}$$

where $\alpha\mathtt{:numeral}$

$$2^{\texttt{nat}} \;+^{\texttt{nat}\to\texttt{nat}\to\texttt{nat}}\; 2^{\texttt{nat}} \;=^{\texttt{nat}\to\texttt{nat}\to\texttt{bool}}\; 4^{\texttt{nat}}$$

$$\Big\downarrow \text{ print}$$

```
(2:nat) + 2 = 4
```

$$\Big\downarrow \text{ parse}$$

$$2^{\texttt{nat}} \;+^{\texttt{nat}\to\texttt{nat}\to\texttt{nat}}\; 2^{\texttt{nat}} \;=^{\texttt{nat}\to\texttt{nat}\to\texttt{bool}}\; 4^{\texttt{nat}}$$

Goal: Calculate a set of annotations that is

(A) **Complete:** reparsing term must not change its type

(B) **Minimal:** annotations must impair readability as little as possible

$$\mathbf{f}^{\mathtt{nat\to int\to bool}}\ \ \mathbf{x}^{\mathtt{nat}}\ \ \mathbf{y}^{\mathtt{int}}$$

type erasure
≈ printing

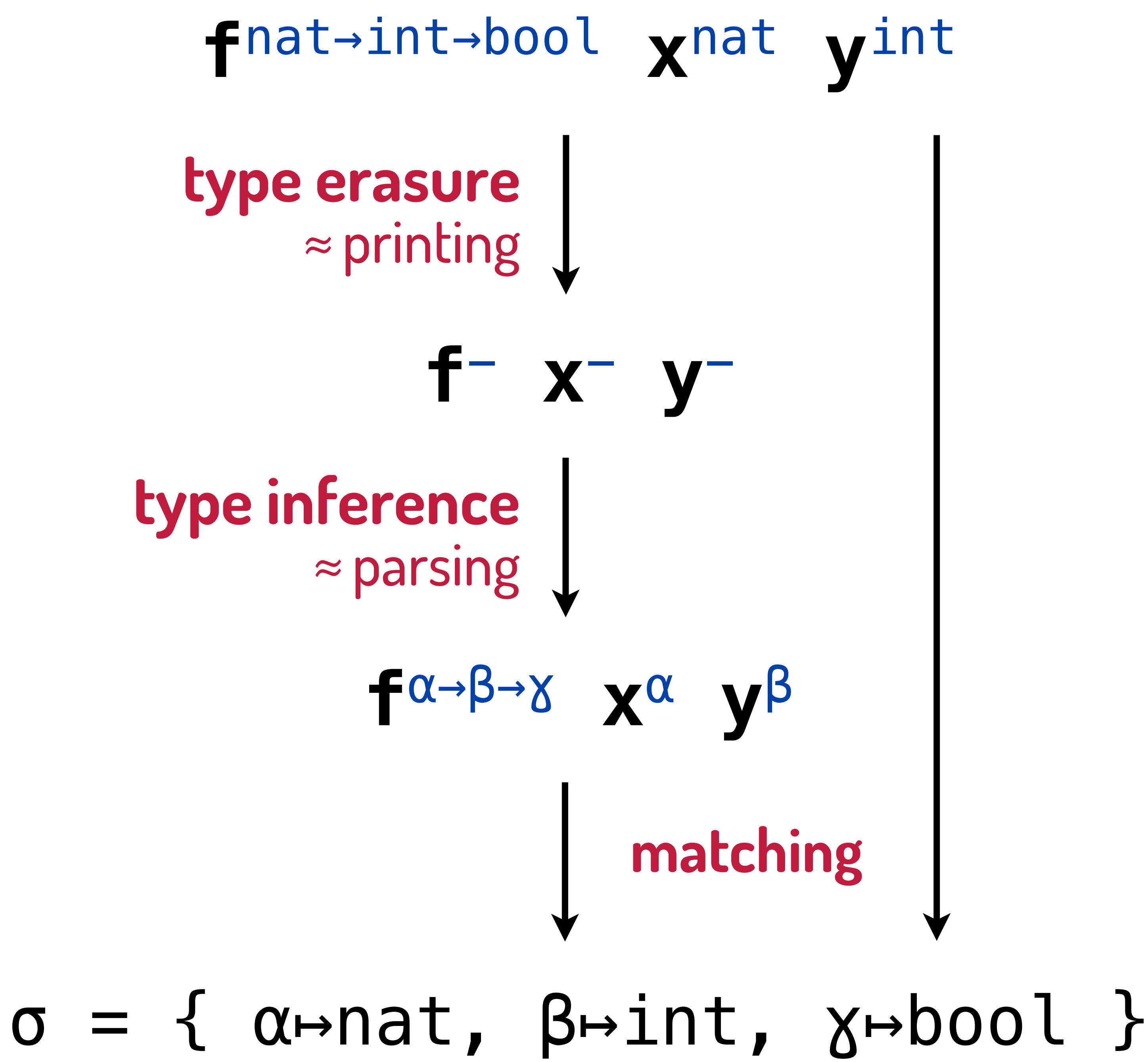
$$\mathbf{f}^{-}\ \ \mathbf{x}^{-}\ \ \mathbf{y}^{-}$$

type inference
≈ parsing

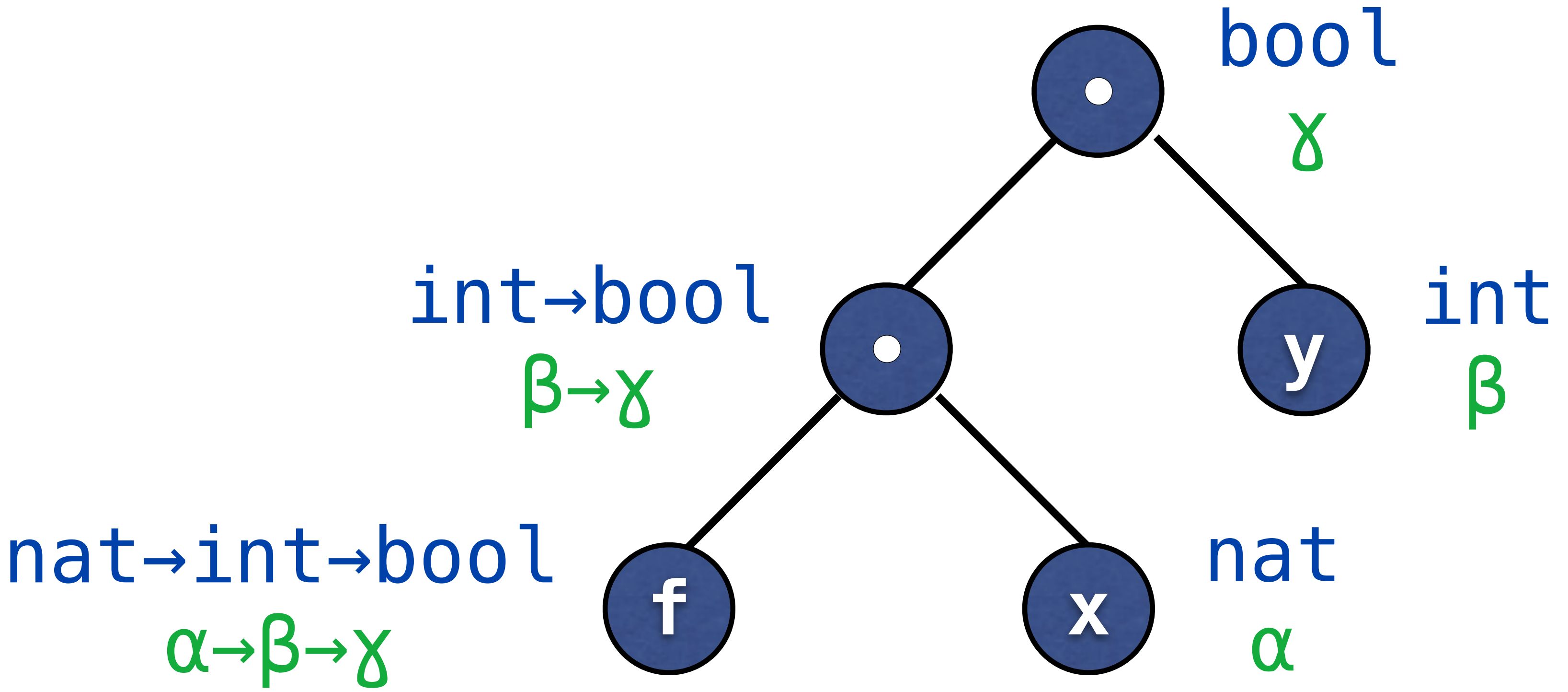
$$\mathbf{f}^{\alpha\to\beta\to\gamma}\ \ \mathbf{x}^{\alpha}\ \ \mathbf{y}^{\beta}$$

matching

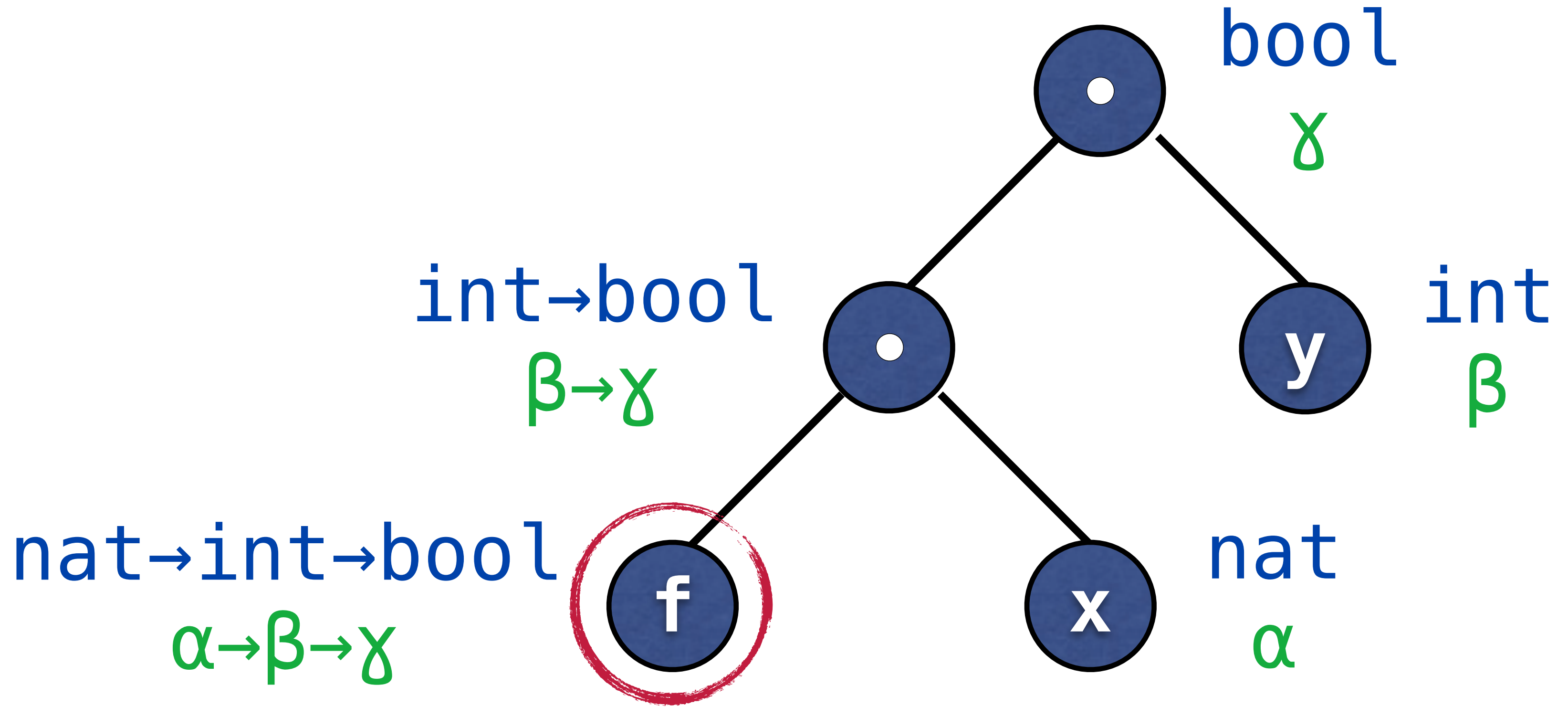
$$\sigma = \{\ \alpha\mapsto \mathtt{nat},\ \beta\mapsto \mathtt{int},\ \gamma\mapsto \mathtt{bool}\ \}$$



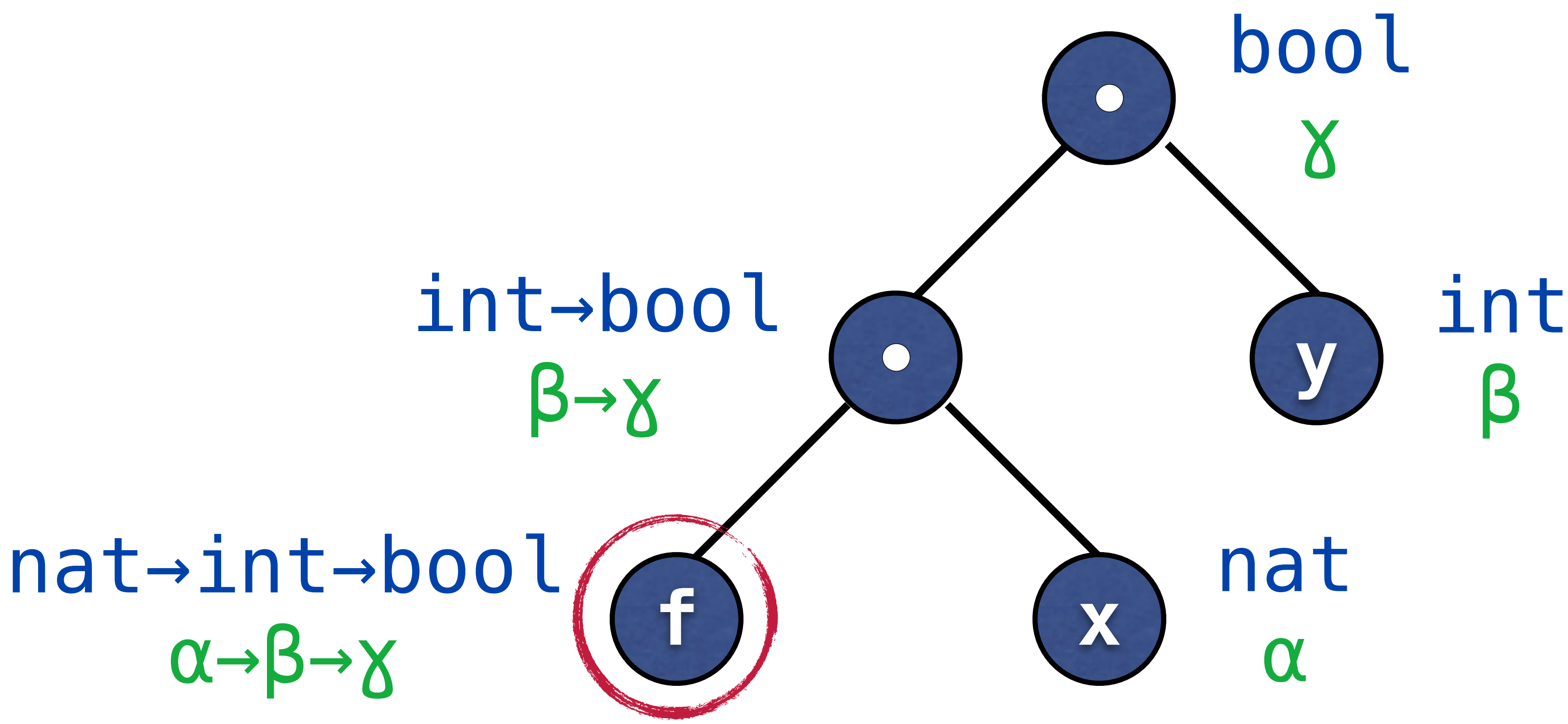
Set of ann. complete IFF it covers Dom(σ)



f x y

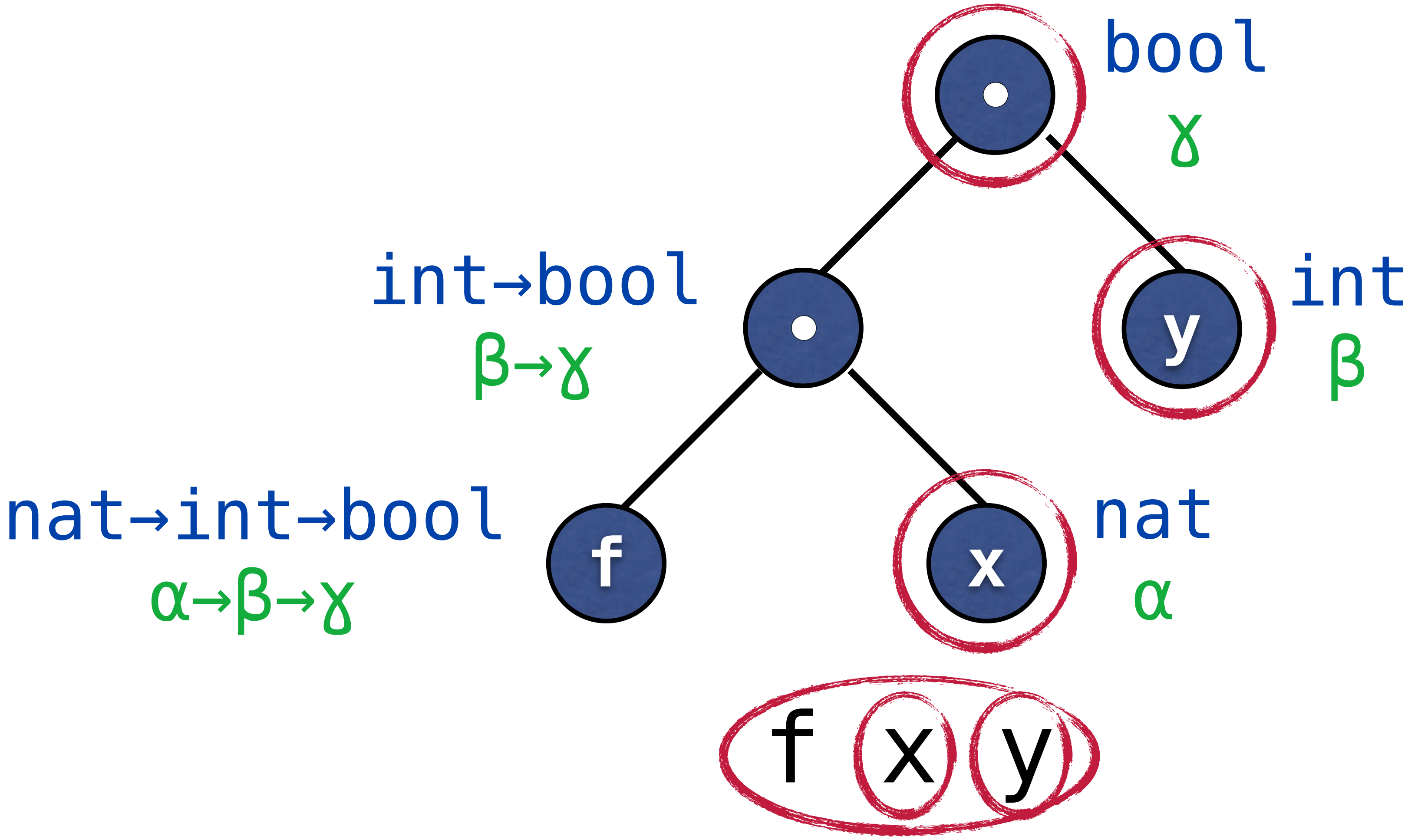


f x y

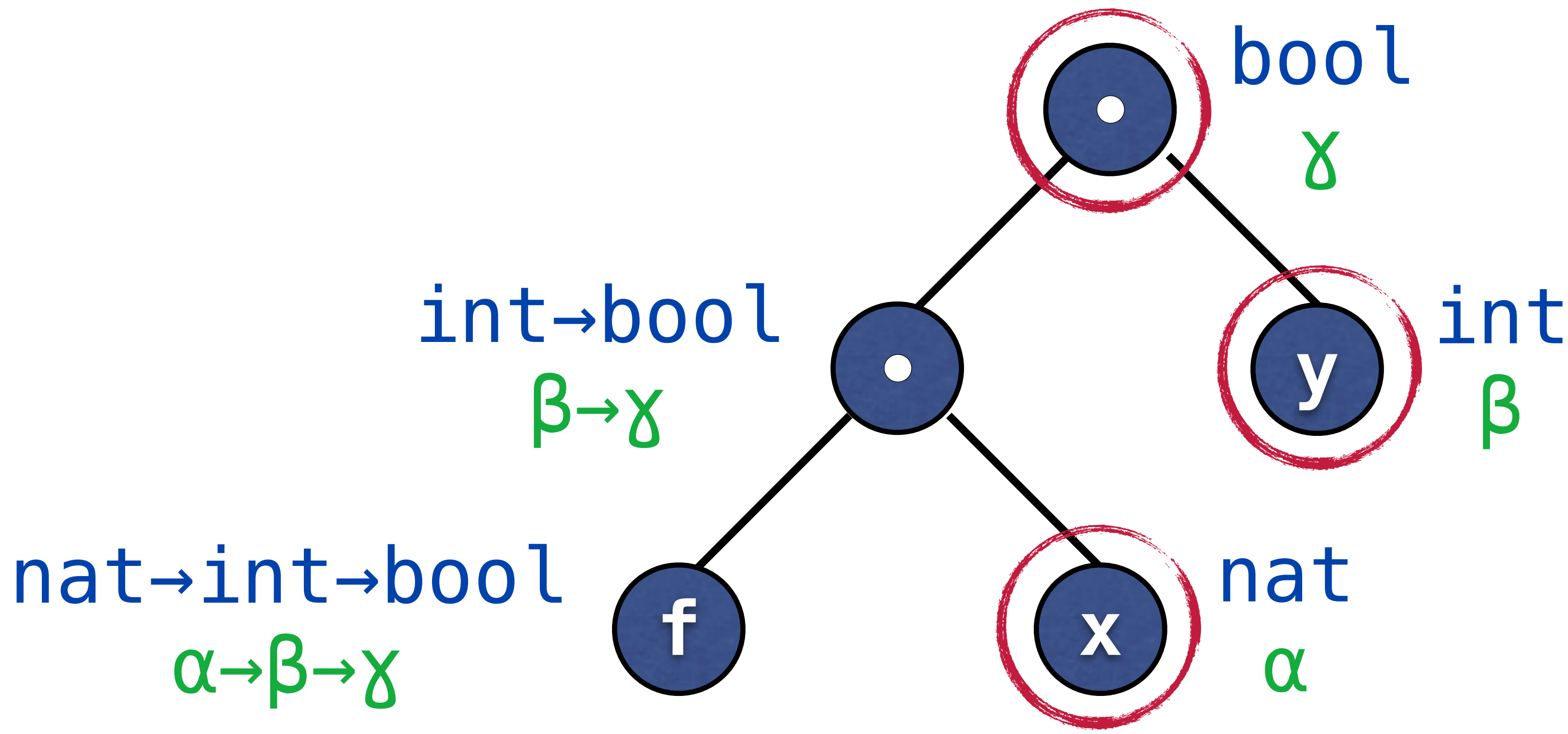


f x y

(f:nat→int→bool) x y



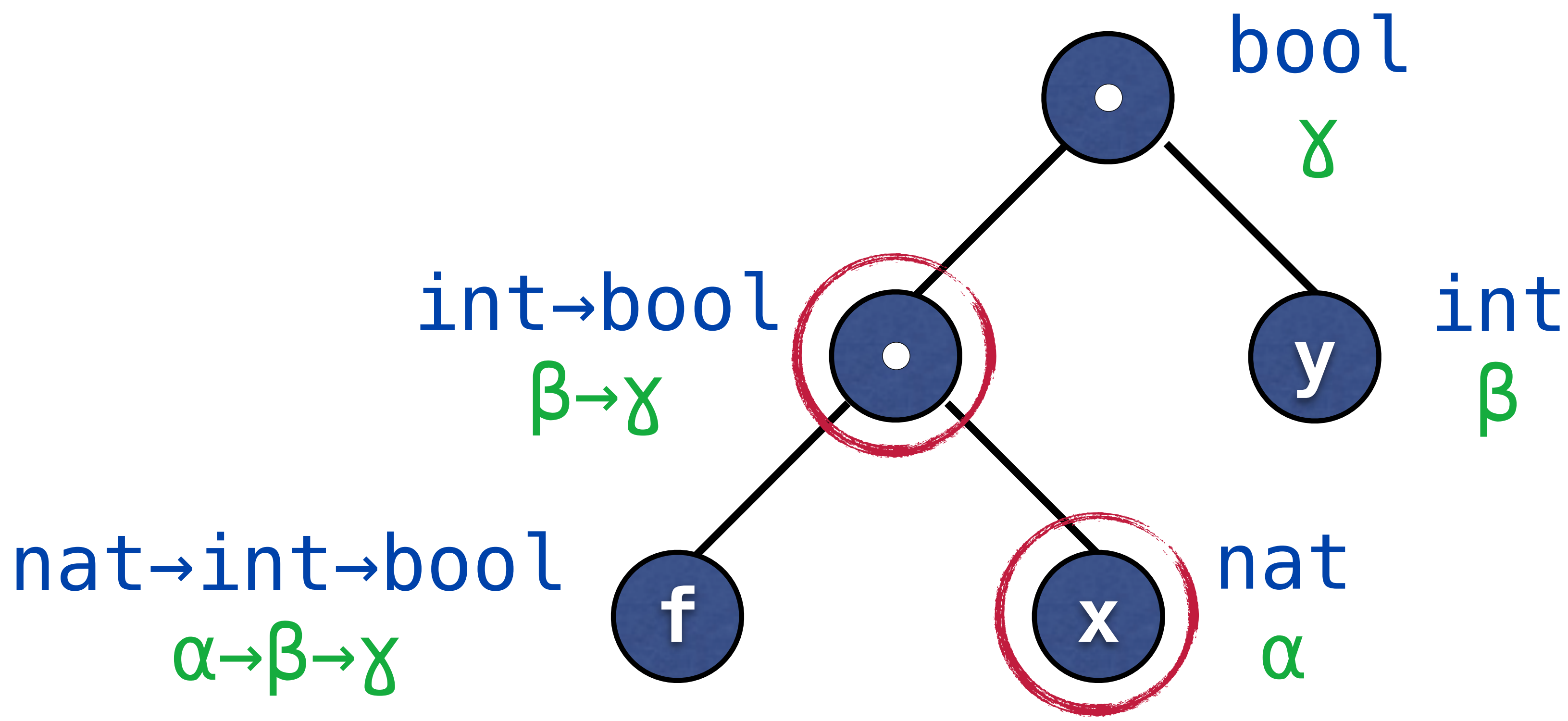
(f:nat→int→bool) x y



f x y

(f:nat→int→bool) x y

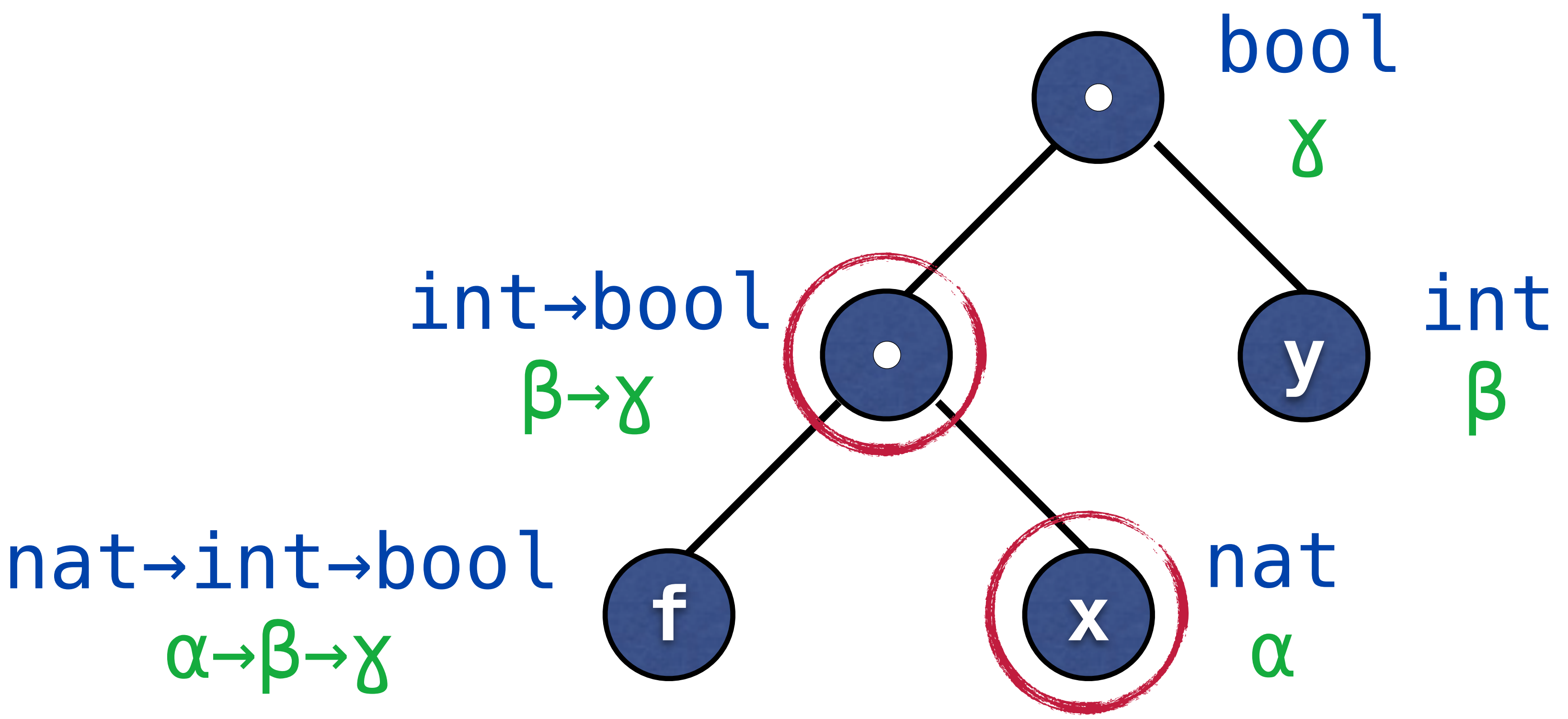
f (x:nat) (y:int) :bool



f x y

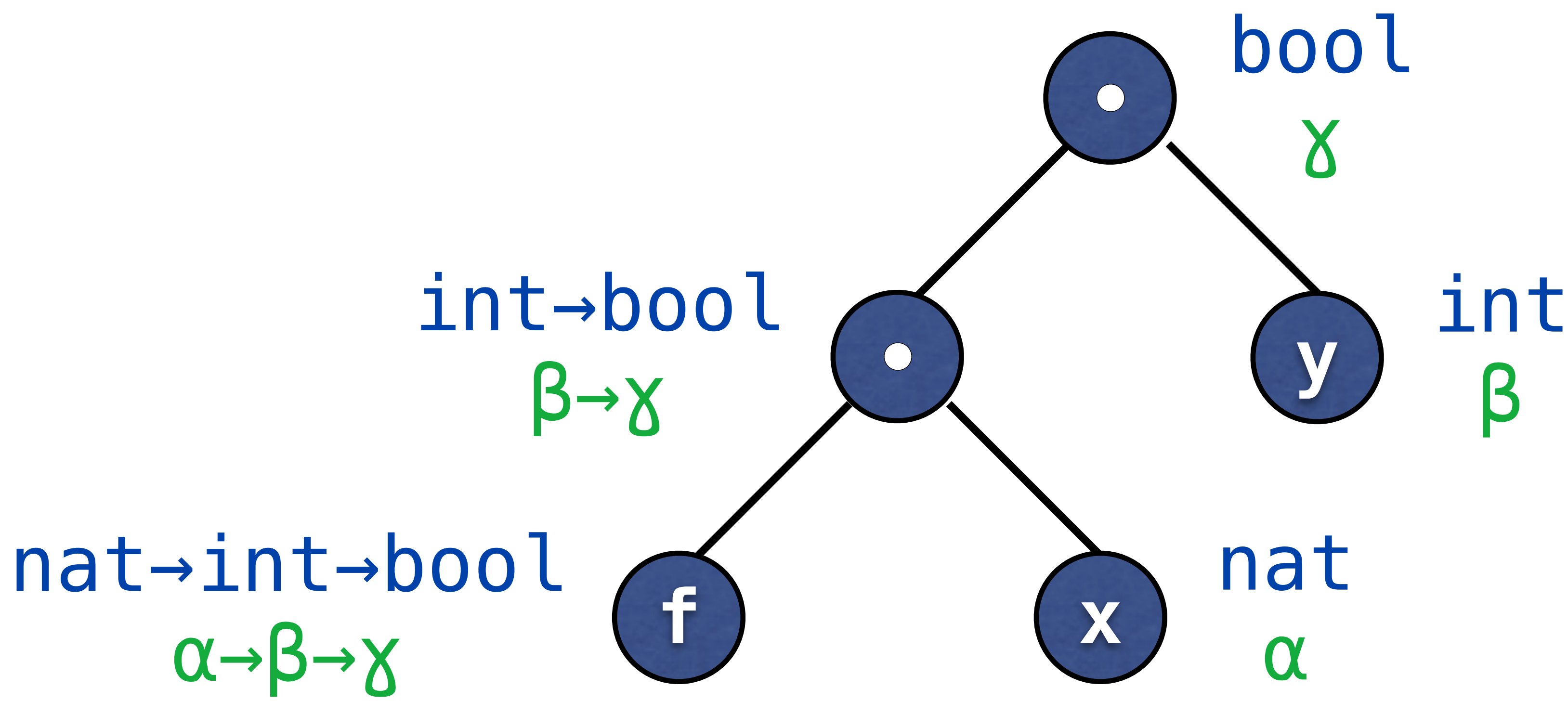
`(f:nat→int→bool) x y`

`f (x:nat) (y:int) :bool`



f x y

```
(f:nat→int→bool) x y
f (x:nat) (y:int) :bool
(f (x:nat) :int→bool) y
```




```
f x y

(f:nat→int→bool) x y

f (x:nat) (y:int) :bool

(f (x:nat) :int→bool) y
```

Which set of annotations is the best?

How do we compute it efficiently?

Which set of annotations is the best?

`cost of t`$^{\tau}$ `:=`

Which set of annotations is the best?

cost of t$^{\tau}$ `:=`

`(size of` τ`,` → small annotations

Which set of annotations is the best?

cost of $\mathtt{t}^{\tau}$ `:=`

`(size of` τ`,` → small annotations

` size of t,` → small annotated terms

Which set of annotations is the best?

cost of t$^{\tau}$:=

```
  (size of τ,             → small annotations
   size of t,             → small annotated terms
   postindex of tτ)       → annotations at the beginning
```

Which set of annotations is the best?

```
cost of tτ :=
  (size of τ,              → small annotations
   size of t,              → small annotated terms
   postindex of tτ)        → annotations at the beginning
```

≤ lexiographically

\+ component-wise

How do we compute it efficiently?

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Instance of **Weighted Set Cover Problem**:

- Finite Universe U → Dom(σ)
- Family S$\subseteq 2^U$ → Possible Annotations

How do we compute it efficiently?

Instance of **Weighted Set Cover Problem**:

- Finite Universe U → Dom(σ)
- Family $S \subseteq 2^U$ → Possible Annotations
- Find $\{U_1,\ldots,U_n\} \subseteq S$ such that
  - $U_1 \cup \ldots \cup U_n = U$ → Completeness
  - cost $\{U_1,\ldots,U_n\}$ minimal → Readability

SCP is **NP-complete** $\implies$ settle for Approximation

Reverse-Greedy Alg. calculates local min:

- start with all annotations
- repeatedly remove the most expensive superfluous annotation

Challenge 4:

Preplay & Compression

Proof Preplay

Generated proofs are only useful if they...

- work
- are reasonably fast

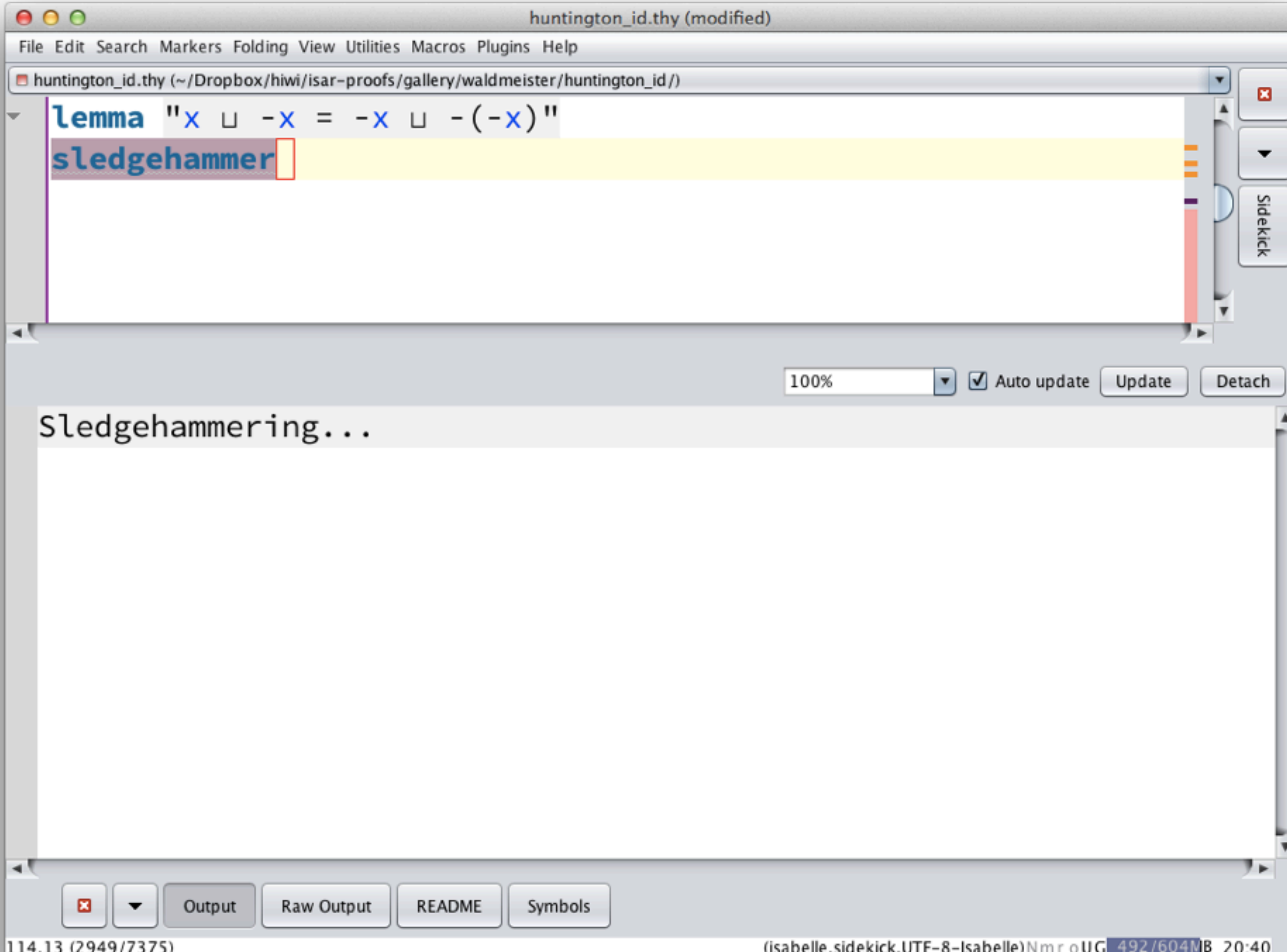
Proof Preplay

Generated proofs are only useful if they...

- work
- are reasonably fast

Let the computer find out!

$\rightarrow$ Present proofs with "preplay" information


huntington_id.thy (modified)
File Edit Search Markers Folding View Utilities Macros Plugins Help
huntington_id.thy (~/Dropbox/hiwi/isar-proofs/gallery/waldmeister/huntington_id/)
lemma "x ⊔ -x = -x ⊔ -(-x)"
sledgehammer
Sidekick
100%
Auto update
Update
Detach
Sledgehammering...
Output
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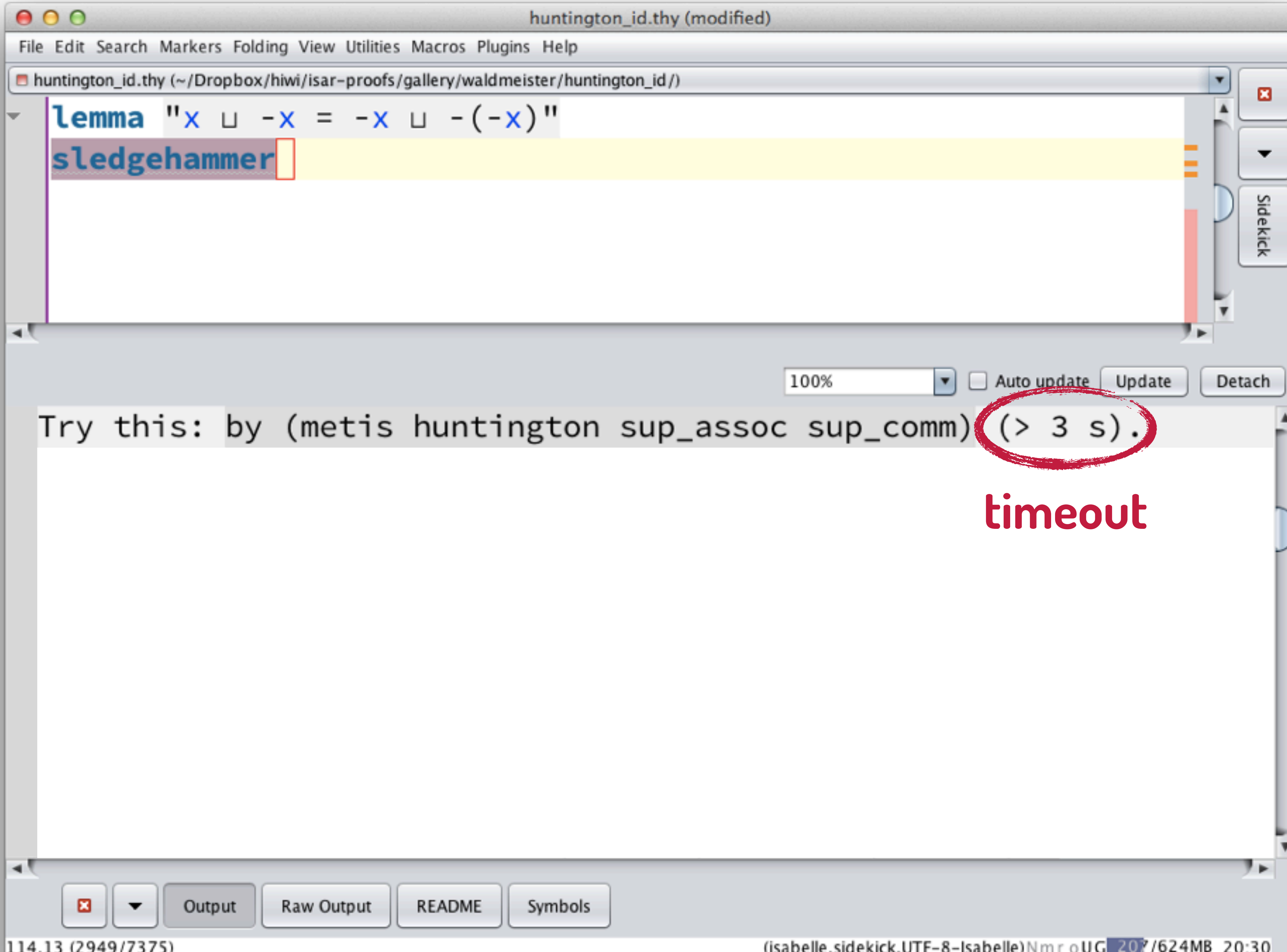
Symbols
114,13 (2949/7375)
(isabelle,sidekick,UTF-8-Isabelle)Nmr oUG 492/604MB 20:40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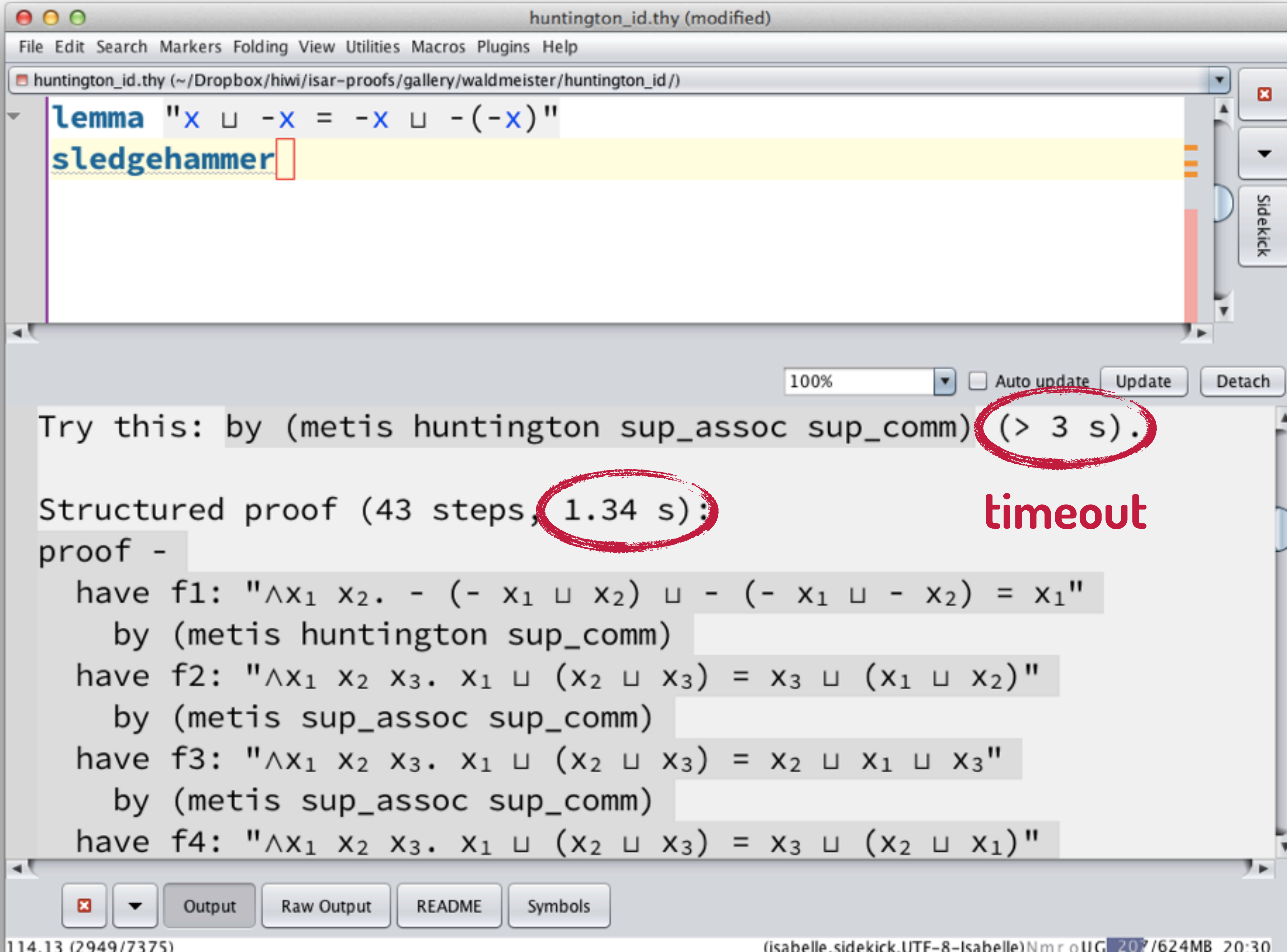
/624MB 20:30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timeout
Structured proof (43 steps, 1.34 s):
proof -
have f1: "⋀x1 x2. - (- x1 ⊔ x2) ⊔ - (- x1 ⊔ - x2) = x1"
by (metis huntington sup_comm)
have f2: "⋀x1 x2 x3. x1 ⊔ (x2 ⊔ x3) = x3 ⊔ (x1 ⊔ x2)"
by (metis sup_assoc sup_comm)
have f3: "⋀x1 x2 x3. x1 ⊔ (x2 ⊔ x3) = x2 ⊔ x1 ⊔ x3"
by (metis sup_assoc sup_comm)
have f4: "⋀x1 x2 x3. x1 ⊔ (x2 ⊔ x3) = x3 ⊔ (x2 ⊔ x1)"
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20:30

Approach A: Feed proof text to Isabelle

+ close to reality

- expensive

- no timings for individual steps

~~**Approach A:** Feed proof text to Isabelle~~

~~+ close to reality~~

~~– expensive~~

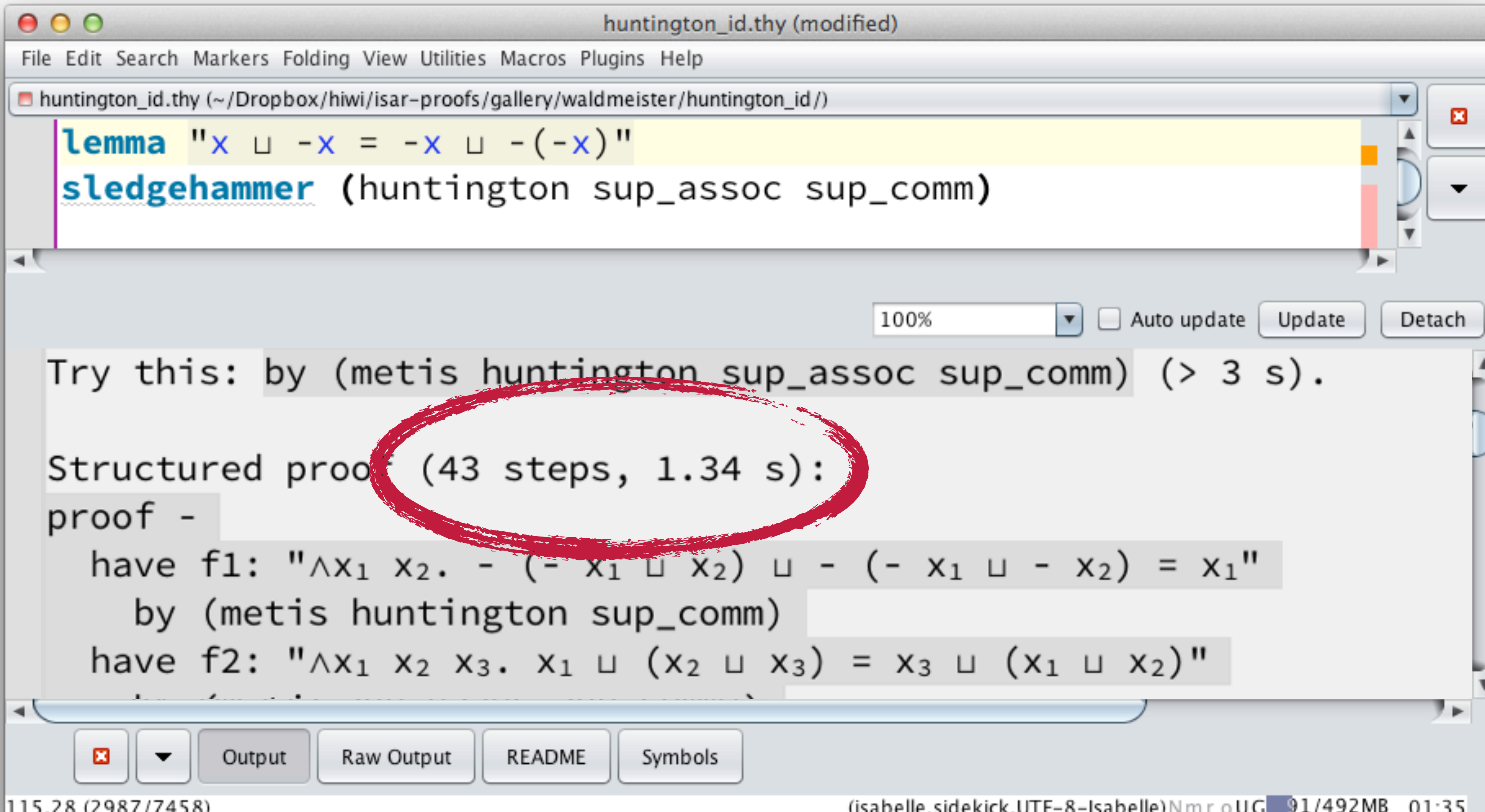
~~– no timings for individual steps~~

Approach B: Simulate replay on ML-level

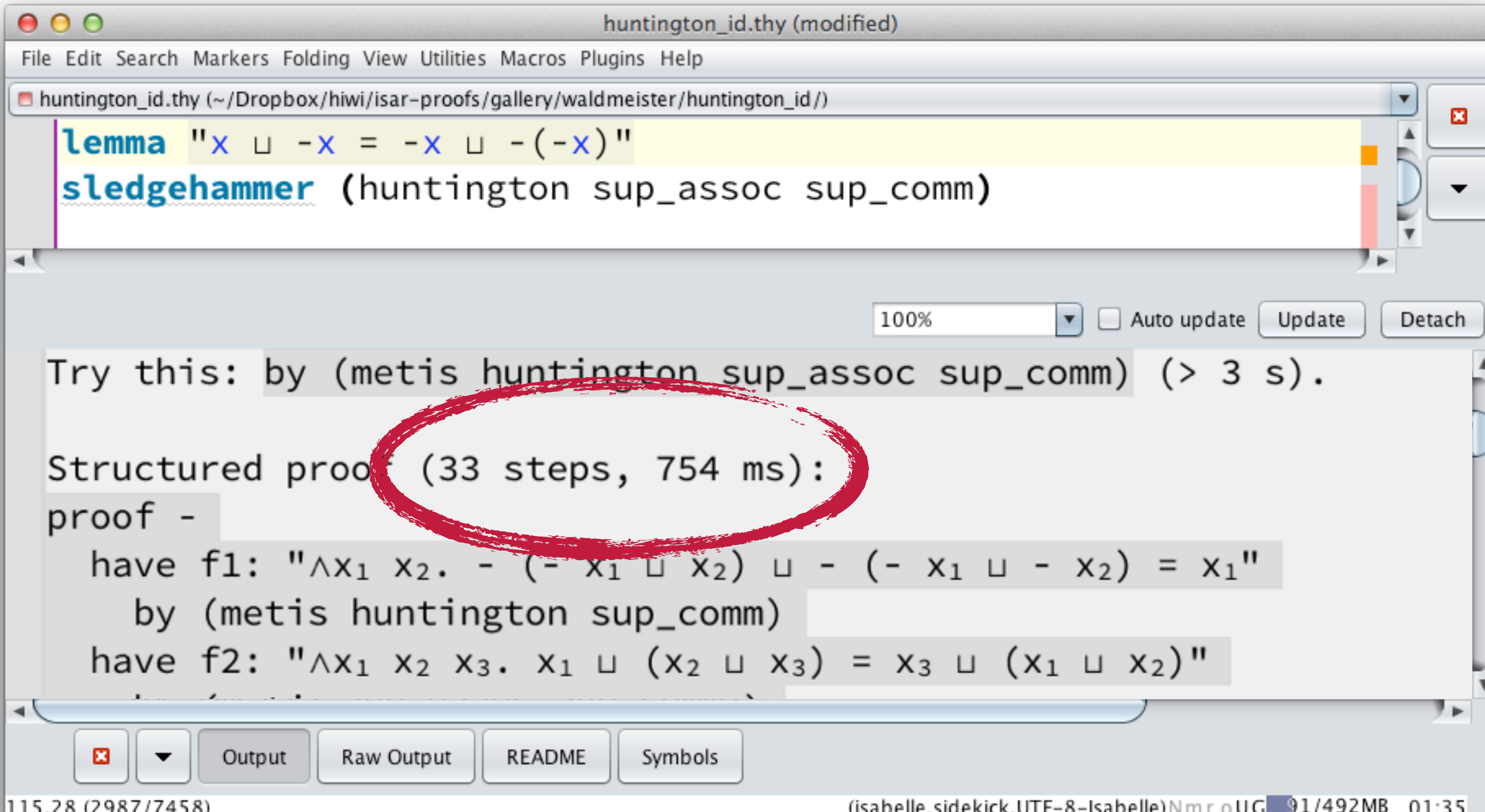
– not the real thing (no printing, no parsing)

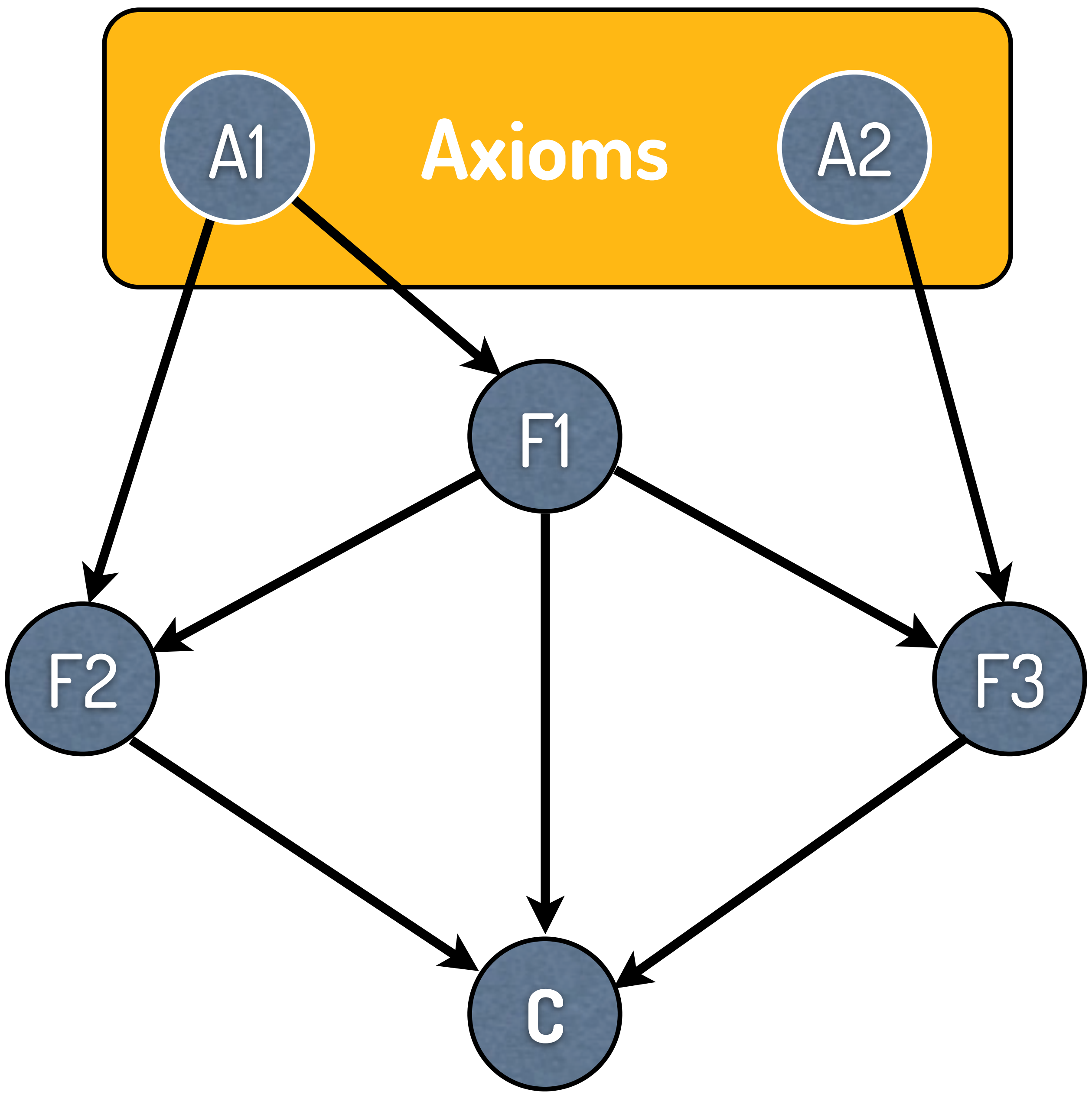
+ timings for each step

Proof Compression



Proof Compression



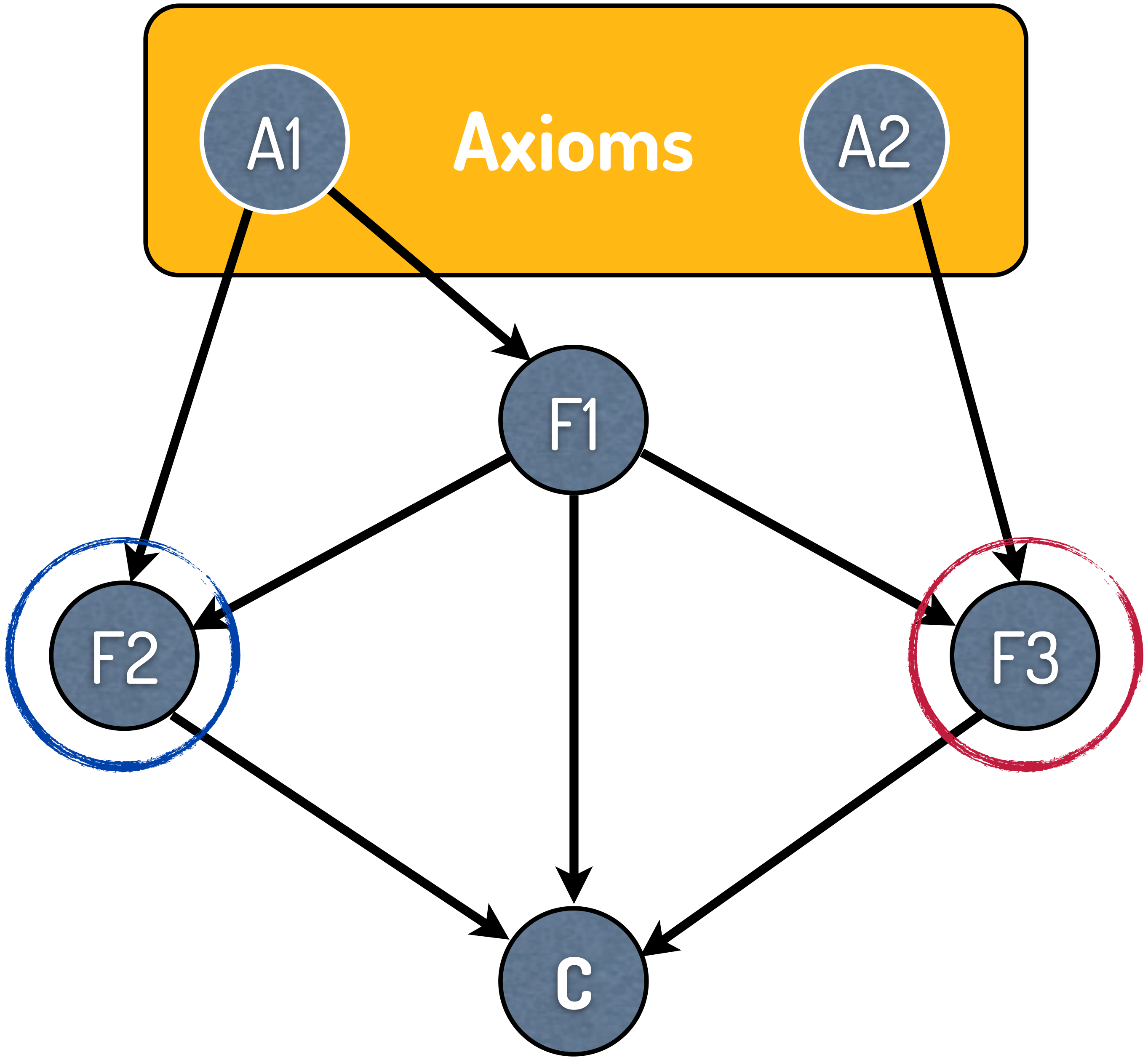


$A1 \vdash F1$

$A1, F1 \vdash F2$

$A2, F1 \vdash F3$

$F1, F2, F3 \vdash C$

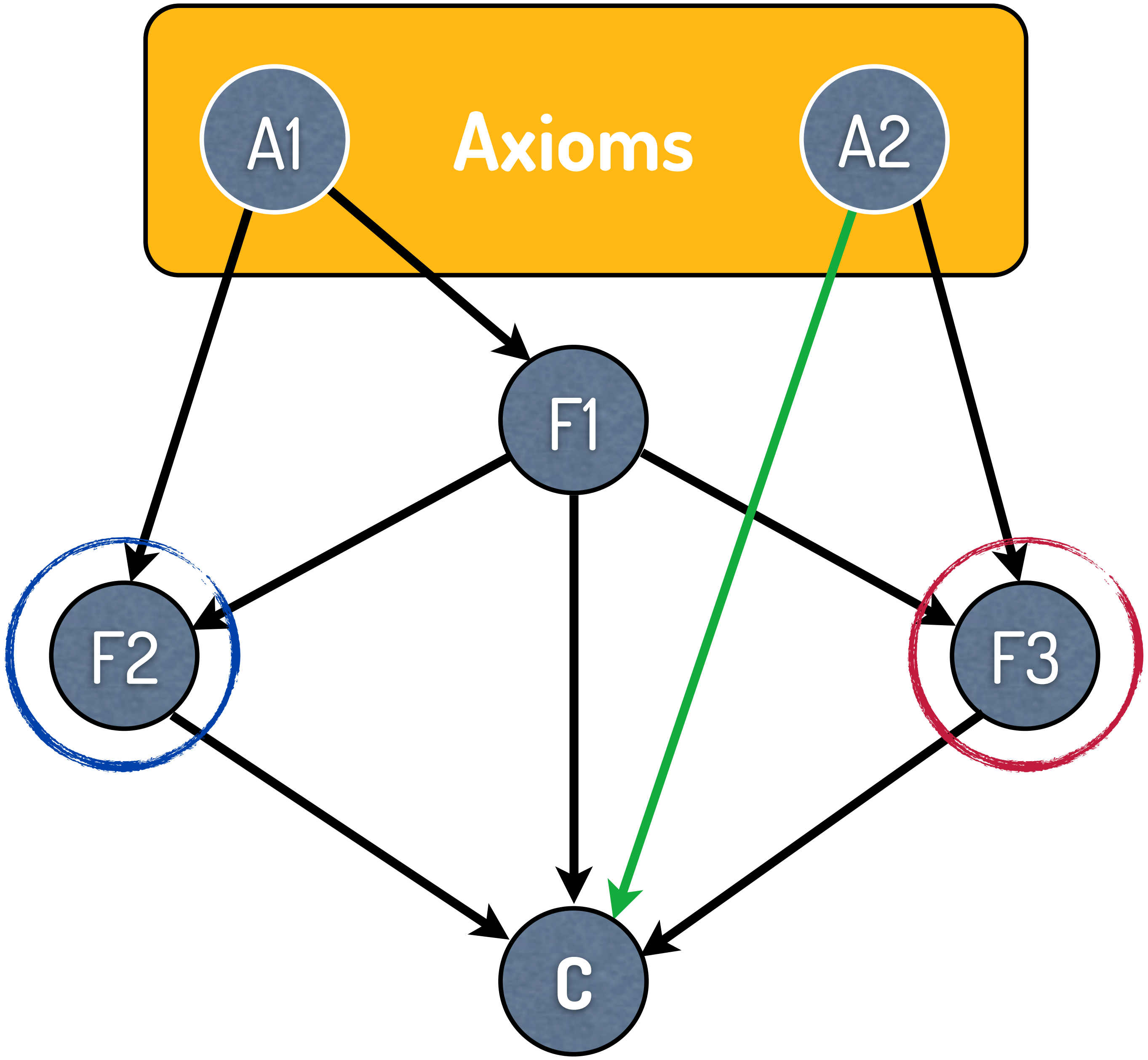


$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$A2, F1 \vdash \boldsymbol{F3}$

$F1, \boldsymbol{F2}, \boldsymbol{F3} \vdash C$

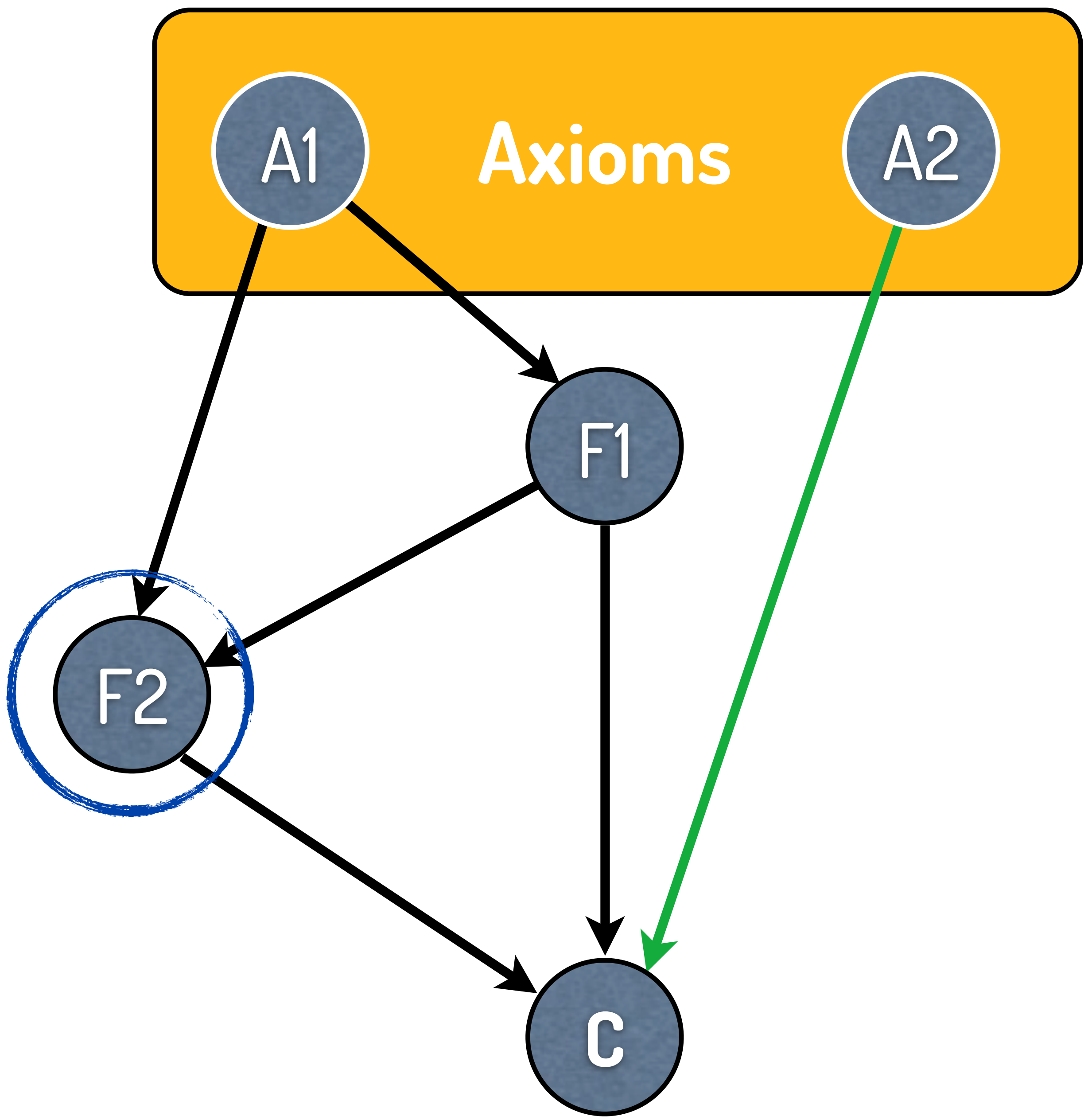


$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$A2, F1 \vdash \boldsymbol{F3}$

$F1, \boldsymbol{F2}, \boldsymbol{F3} \vdash C$



A1 ⊢ F1

A1,F1 ⊢ ***F2***

A2,F1 ⊢ ***F3***

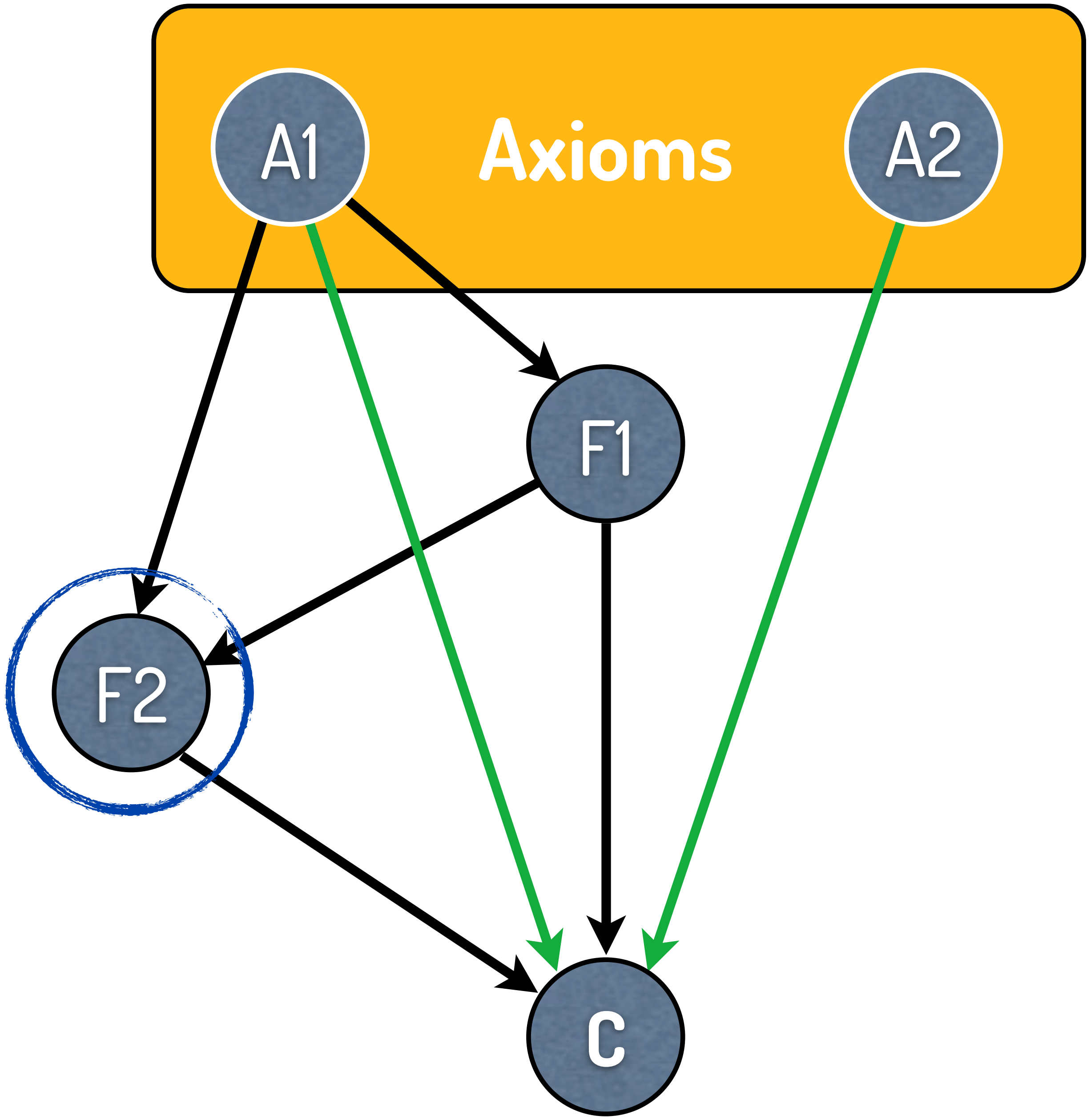
F1,***F2***,***F3*** ⊢ C

↓

A1 ⊢ F1

A1,F1 ⊢ ***F2***

F1,***F2***,A2 ⊢ C



$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$A2, F1 \vdash \boldsymbol{F3}$

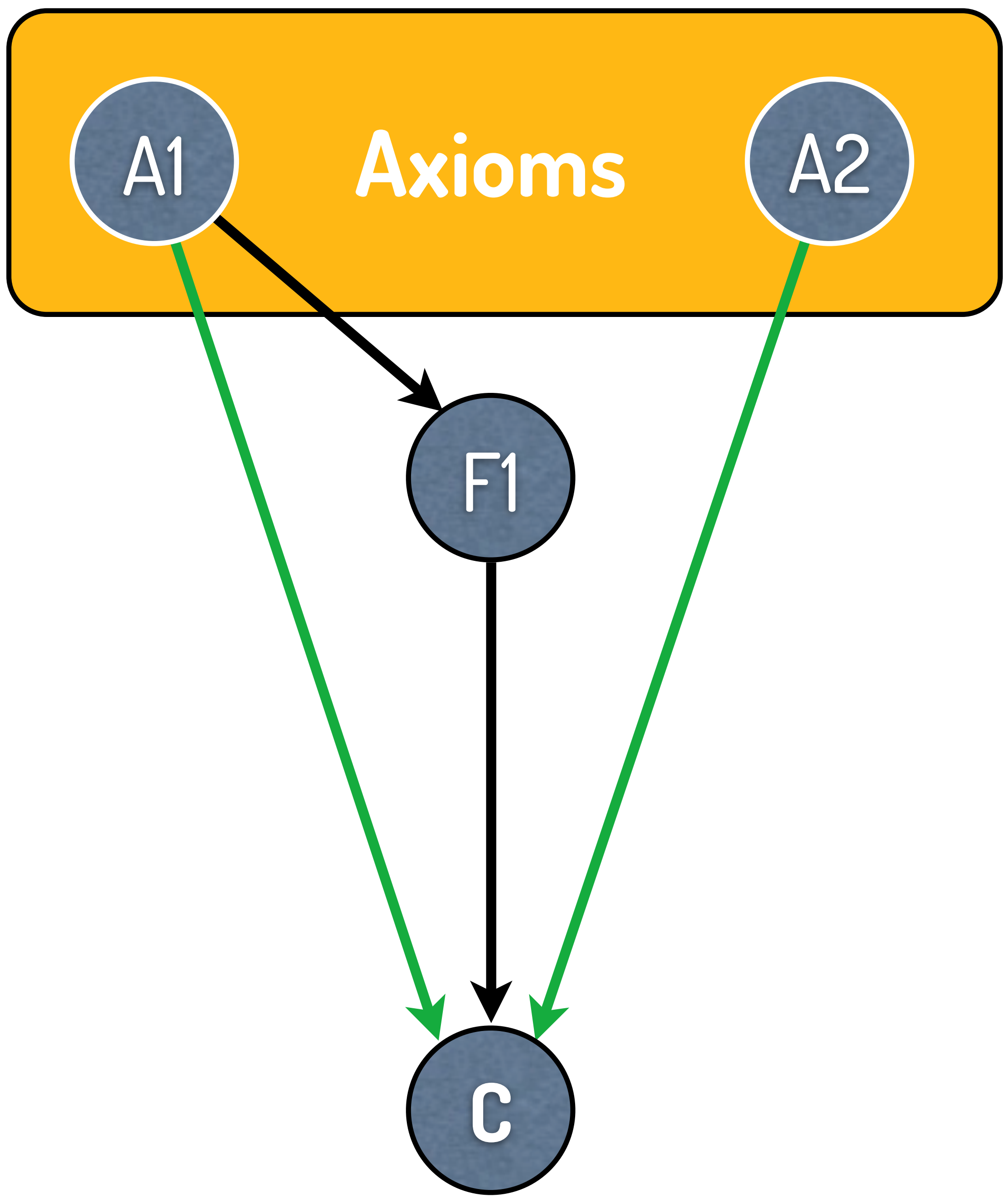
$F1, \boldsymbol{F2}, \boldsymbol{F3} \vdash C$

$\Downarrow$

$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$F1, \boldsymbol{F2}, A2 \vdash C$



$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$A2, F1 \vdash \boldsymbol{F3}$

$F1, \boldsymbol{F2}, \boldsymbol{F3} \vdash C$

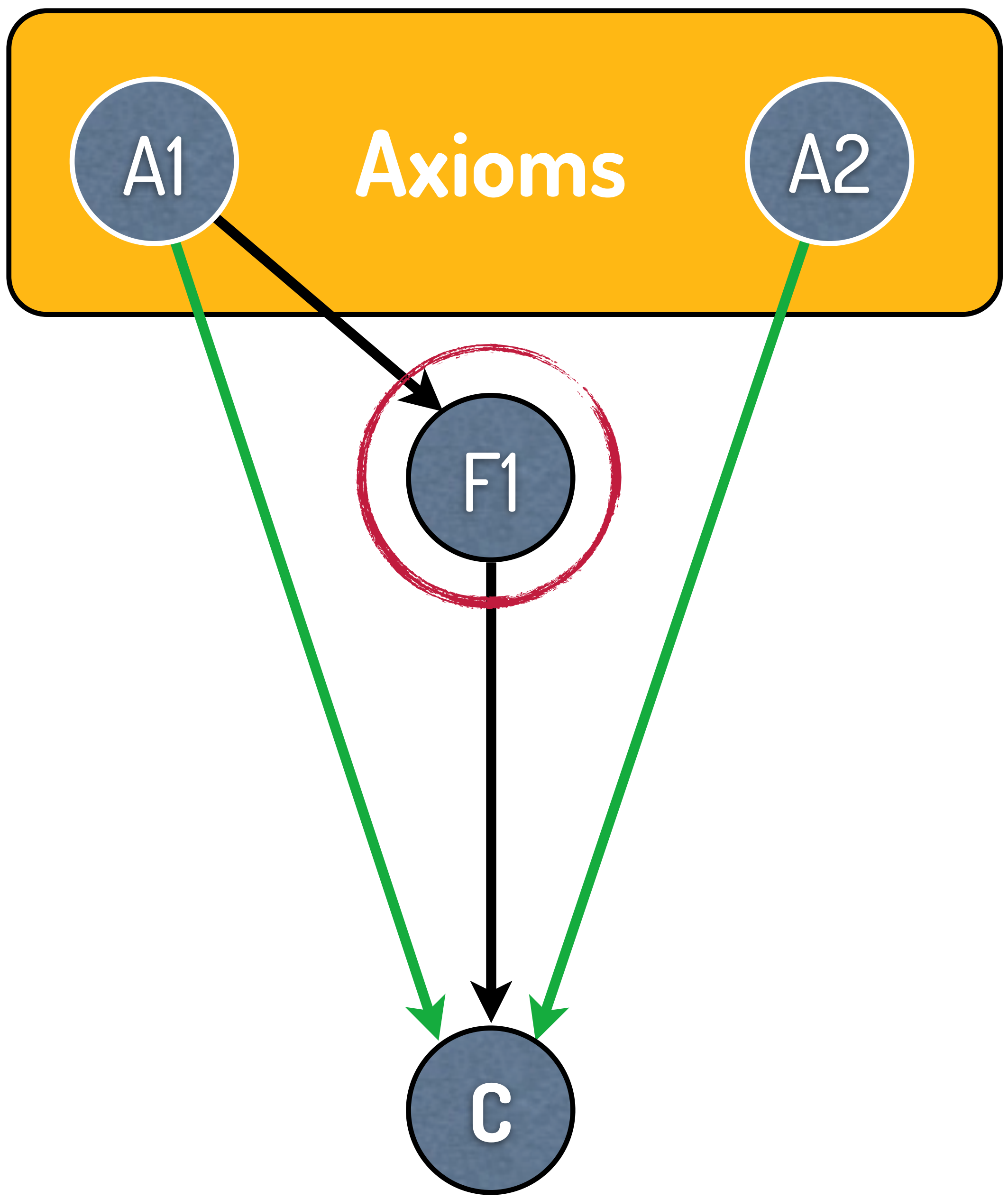
$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$F1, \boldsymbol{F2}, A2 \vdash C$

$A1 \vdash F1$

$F1, A1, A2 \vdash C$



$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$A2, F1 \vdash \boldsymbol{F3}$

$F1, \boldsymbol{F2}, \boldsymbol{F3} \vdash C$

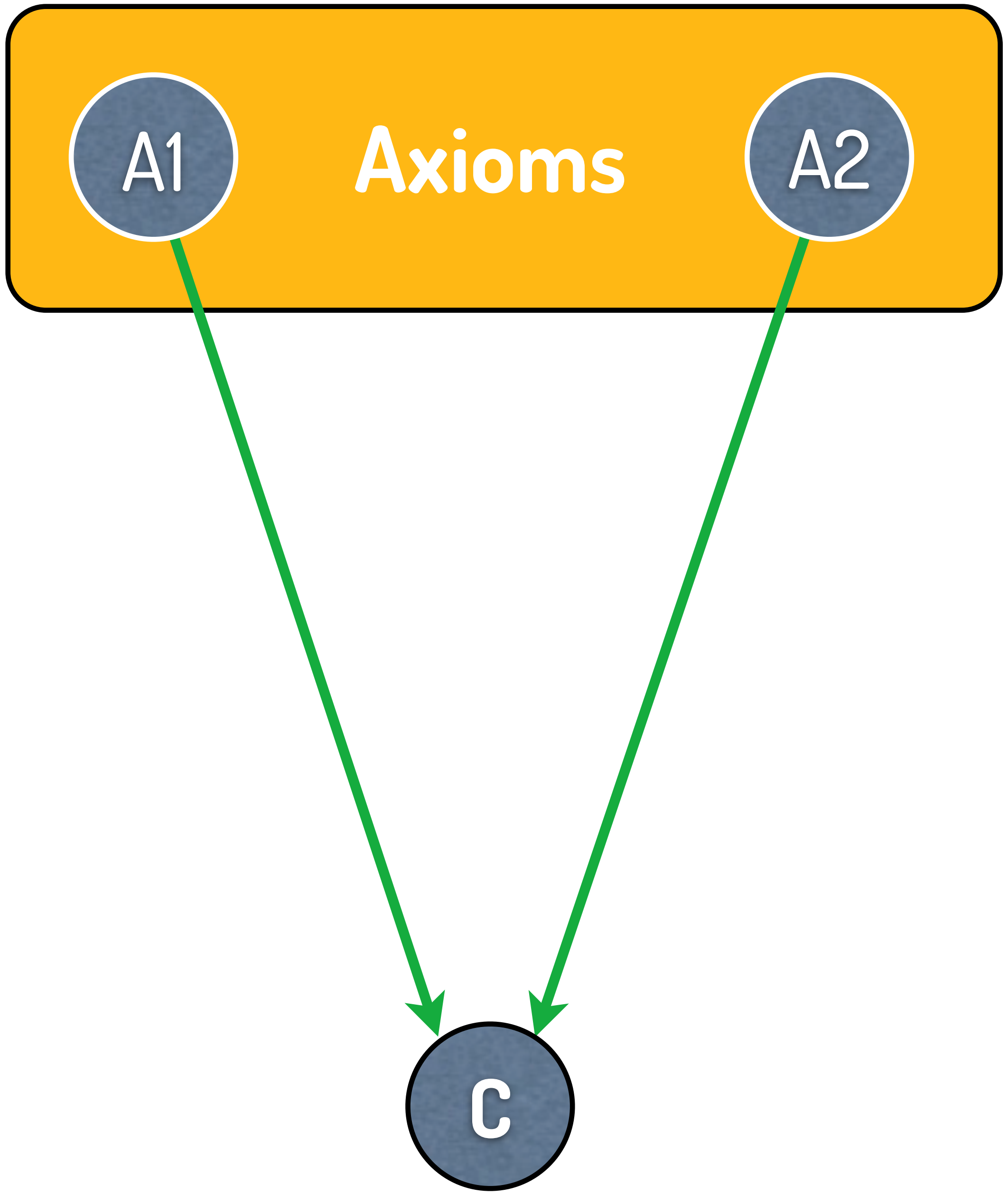
$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$F1, \boldsymbol{F2}, A2 \vdash C$

$A1 \vdash \boldsymbol{F1}$

$\boldsymbol{F1}, A1, A2 \vdash C$



$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$A2, F1 \vdash \boldsymbol{F3}$

$F1, \boldsymbol{F2}, \boldsymbol{F3} \vdash C$

$A1 \vdash F1$

$A1, F1 \vdash \boldsymbol{F2}$

$F1, \boldsymbol{F2}, A2 \vdash C$

$A1 \vdash \boldsymbol{F1}$

$\boldsymbol{F1}, A1, A2 \vdash C$

$A1, A2 \vdash C$

$$\left.\begin{array}{l} \mathtt{A2,F1} \vdash \boldsymbol{\mathit{F3}} \\ \mathtt{F1,F2,}\boldsymbol{\mathit{F3}} \vdash \mathtt{P} \end{array}\right\} \quad \mathtt{F1,F2,A2} \vdash \mathtt{P}$$

Does merger save time? → Preplay

Have we reached a given compression factor?

Robust, Semi-Intelligible Isabelle Proofs from ATP Proofs

Steffen Juilf Smolka
Jasmin Christian Blanchette